Derivation for CPB Hamiltonian

$$Q = CV \Rightarrow -2e\langle n\rangle = C_g(V - V_g) + C_j V$$

$$\Rightarrow V(n) = \frac{C_g V_g - 2ne}{C_g + C_j}$$

What energy is required to put extra charge Δn onto the island?

$$E(n+\Delta n) - E(n) = -2e\int_0^{\Delta n} V(n+n')\,dn'$$

$$= -2e\int_0^{\Delta n} \frac{C_g V_g - 2(n+n')e}{C_g + C_j}\,dn'$$

$$= -2e\left\{\frac{C_g V_g}{C_g + C_j}\Delta n - \frac{2ne}{C_g + C_j}\Delta n - \frac{\Delta n^2 e}{C_g + C_j}\right\}$$

$$= \frac{2e^2\Delta n^2 + 4e^2 n\Delta n - 2eC_g V_g\Delta n}{C_g + C_j}$$

Total charging energy :-

$$E(0+n) - E(0) = \frac{2e^2 n^2 + 0 - 2eC_g V_g n}{C_g + C_j}$$

Now tidy this up by setting $E(0)$ to something sensible :-

$$E(n) - E(0) = \frac{e^2}{2(C_g + C_j)}\Big\{\underbrace{4n^2 - 4\frac{C_g V_g}{e}n}_{\text{complete the square here...}}\Big\}$$

$(n - n_g)^2 - n_g^2$ for $n_g = \dfrac{C_g V_g}{2e}$

$$\ldots = 4E_c(n - n_g)^2 - \underbrace{\left(\frac{C_g V_g}{2e}\right)^2\left(\frac{2e^2}{C_g + C_j}\right)}_{\text{Choose this for } E(0)}$$

$$\boxed{E_c = \frac{e^2}{2(C_g + C_j)} = \frac{e^2}{2C_\Sigma}}$$

Now Hamiltonian is :-

$$\mathcal{H}_{el} = \sum_n \Big[4E_c(n - n_g)^2\,\underbrace{|n\rangle\langle n|}_{\text{projection operators.}}\Big]$$

Josephson coupling

Phase / charge conjugate variables

$$[\hat{\varphi}, \hat{q}] = i\hbar \quad \hat{n} = \frac{1}{e}\hat{q}$$
$$[\hat{\delta}, \hat{n}] = i \quad \hat{\delta} = \frac{e}{\hbar}\hat{\varphi}$$

$$\hat{H}_J = -E_J \cos(\varphi_1 - \varphi_2)\, |\varphi_1, \varphi_2\rangle\langle\varphi_1, \varphi_2|$$

zero phase diff. is preferred state, whereas π phase diff. is high energy

Transform to number basis with identities :-

$$\hat{H}_J = -\sum_{n_1,n_2,n_3,n_4} |n_1, n_2\rangle\langle n_1, n_2 | \varphi_1, \varphi_2\rangle E_J \cos(\varphi_1 - \varphi_2) \times \langle \varphi_3, \varphi_4 | n_3, n_4\rangle\langle n_3, n_4|$$

overlap integrals of n, φ :-

$$\ldots = -E_J \sum_{n_1,n_2,n_3,n_4} \int_0^{2\pi} \frac{d\varphi_1}{2\pi} e^{-i(n_1-n_3)\varphi_1} \int_0^{2\pi} \frac{d\varphi_2}{2\pi} e^{-i(n_2-n_4)\varphi_2}$$
$$\times \underbrace{\frac{1}{2}\left(e^{i(\varphi_1-\varphi_2)} + e^{i(\varphi_2-\varphi_1)}\right)}_{\cos(\Delta\varphi)} \quad |n_1, n_2\rangle\langle n_3, n_4|$$

integrals are delta functions

$$\int_0^{2\pi} \frac{d\varphi_1}{2\pi} \left\{ e^{-i(n_1 - n_3 - 1)\varphi_1} + e^{-i(n_1 - n_3 + 1)\varphi_1} \right\} \times$$

$$\int_0^{2\pi} \frac{d\varphi_2}{2\pi} \left\{ e^{-i(n_2 - n_4 + 1)\varphi_2} + e^{-i(n_2 - n_4 - 1)\varphi_2} \right\} \times (\ldots)$$

Hence $n_3 = n_1 \pm 1$

$n_4 = n_2 \pm 1$

Since one side of the junction is simply the ground, forget about it ...

$$\hat{H}_J = -\frac{E_J}{2}\left[\,|n\rangle\langle n+1| + |n+1\rangle\langle n|\,\right]$$

Two level approximation

$$\hat{H} = \sum_n \left\{ 4E_C (\hat{n}-n_g)^2 \, |n\rangle\langle n| - \frac{E_J}{2}\left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right) \right\}$$

Restrict to $n = 0, 1$

$$\hat{H} = 4E_C \Big\{ \{ n_g^2 \, |0\rangle\langle 0| + (1-n_g)^2 |1\rangle\langle 1| \} - \{ n_g^2 + \tfrac{1}{2}(1-2n_g) \} \mathbb{1} \Big\} - \frac{E_J}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)$$

← choose a new zero point energy halfway between $|0\rangle$ and $|1\rangle$ energies.

$$= 4E_C \left\{ -\left(\frac{1-2n_g}{2}\right)|0\rangle\langle 0| + \left(\frac{1-2n_g}{2}\right)|1\rangle\langle 1| \right\} - \frac{E_J}{2}\left\{ |1\rangle\langle 0| + |0\rangle\langle 1| \right\}$$

$$\hat{H} = \frac{1}{2}\begin{pmatrix} -E_{el} & -E_J \\ -E_J & +E_{el} \end{pmatrix} \quad \text{where } E_{el} = 4E_C(1-2n_g)$$

$$\hat{H} = -\frac{E_{el}}{2}\hat{\sigma}_z - \frac{E_J}{2}\hat{\sigma}_x \quad //$$

identical Hamiltonian to a spin 1/2 particle in a B-field in direction $\begin{pmatrix} E_J \\ 0 \\ E_{el} \end{pmatrix} \left(\times \frac{1}{g\mu}\right)$ (for correct magnitude)

Eigenvectors can be written neatly in terms of a parameter θ :-

$$|g\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)\,|1\rangle$$
$$|e\rangle = \cos(\theta/2)\,|1\rangle - \sin(\theta/2)\,|0\rangle$$

$$\theta = \tan^{-1}\left(\frac{E_J}{E_{el}}\right)$$

$$\hat{n} = \tfrac{1}{2}(1-\hat{\sigma}_z) \Rightarrow \langle g|\hat{n}|g\rangle = \sin^2(\theta/2) \quad //$$
$$\langle e|\hat{n}|e\rangle = \cos^2(\theta/2) \quad //$$