III/2

$\Psi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle \qquad H = -\vec{\mu}\cdot\vec{B} = -\mu\,\vec{\sigma}\cdot\vec{B}$

∘) before B_x - field is applied, nothing changes ~~to~~ with the spin

$$H_{t<t_0} = -\mu\,\sigma_z B_z$$

$$e^{-\frac{i\mu B_z t}{\hbar}\sigma_z} |\downarrow\rangle = e^{\frac{i}{\hbar}\mu B_z t} |\downarrow\rangle = e^{i\frac{\omega}{2}t} |\downarrow\rangle$$

⇒ only global phase factor

∘) at time t_0 B_x - field is switched on:

$$H = -\mu(\sigma_x B_x + \sigma_z B_z)$$

z-field is still there!

→ approximation: $B_z \ll B_x$

$$\Rightarrow H = -\mu\,\sigma_x B_x$$

evolution: $e^{-\frac{i}{\hbar}\overbrace{\mu B_x}^{\omega/2} t\sigma_x} |\downarrow\rangle =$

$$\left(\cos\frac{\omega t}{2}\sigma_x - i\sin\frac{\omega t}{2}\sigma_x\right)|\downarrow\rangle =$$

$$= \left[\left(1 - \frac{1}{2}\left(\frac{\omega t}{2}\right)^2\sigma_x^2 + \frac{1}{4!}\left(\frac{\omega t}{2}\right)^4\sigma_x^4 + \dots\right) - i\left(\frac{\omega t}{2}\sigma_x - \left(\frac{\omega t}{2}\right)^3\sigma_x^3 + \dots\right)\right]|\downarrow\rangle$$

with $\sigma_x^2 = \mathbb{1}$

$$= \left(\mathbb{1}\cdot\cos\frac{\omega t}{2} - i\sigma_x\sin\frac{\omega t}{2}\right)|\downarrow\rangle =$$

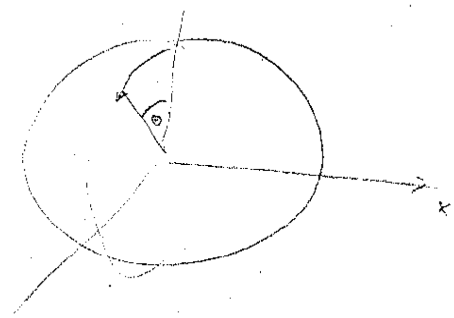
$$= \begin{pmatrix} \cos\omega t/2 & -i\sin\frac{\omega t}{2} \\ -i\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i\sin\frac{\omega t}{2} \\ \cos\frac{\omega t}{2} \end{pmatrix}$$

bring to standard form

$$= e^{-i\frac{\pi}{2}}\begin{pmatrix} \sin\omega t/2 \\ e^{i\frac{\pi}{2}}\cos\frac{\omega t}{2} \end{pmatrix} \hat{=} \sin\omega t/2\,|\uparrow\rangle + e^{i\frac{\pi}{2}}\cos\frac{\omega t}{2}\,|\downarrow\rangle$$

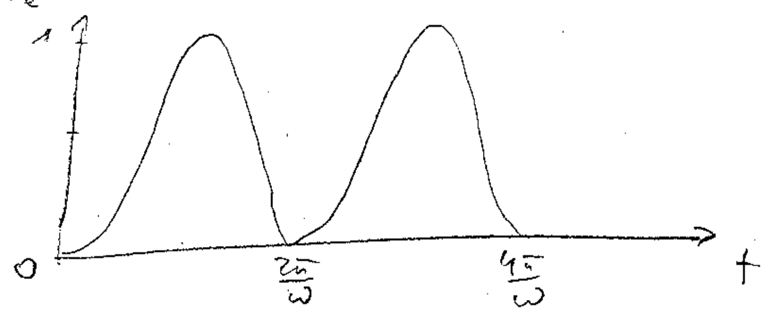
$\Rightarrow$ polar representation: $\theta = \frac{\omega t}{2}$

$\varphi = \frac{\pi}{2}$



excited state population:

$$\sin^2\frac{\omega t}{2} = \frac{1}{2}(1 - \cos\omega t)$$



III/1 ~~Alternative~~

Schrödinger Equation in rotating frame

$$H|\psi\rangle = i\hbar|\dot{\psi}\rangle \qquad |\psi\rangle \to |\phi\rangle = U|\psi\rangle$$

$$U\cdot \quad HU^\dagger|\phi\rangle = i\hbar\frac{d}{dt}\left(U^\dagger|\phi\rangle\right)$$

$$UHU^\dagger|\phi\rangle = i\hbar\left(U\dot{U}^\dagger|\phi\rangle + \overbrace{UU^\dagger}^{\mathbb{1}}|\dot{\phi}\rangle\right)$$

$$\Big(\underbrace{UHU^\dagger - i\hbar U\dot{U}^\dagger}_{\tilde{H}}\Big)|\phi\rangle = i\hbar|\dot{\phi}\rangle$$

$\tilde{H}$:

$$U\sigma_z U^\dagger = \sigma_z$$

$$U\sigma_x U^\dagger = \begin{pmatrix} e^{i\tilde{\omega}t/2} & 0 \\ 0 & e^{-i\tilde{\omega}t/2}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} e^{-i\tilde{\omega}t/2} & 0 \\ 0 & e^{+i\tilde{\omega}t/2}\end{pmatrix}$$

$$= \begin{pmatrix} e^{i\tilde{\omega}t/2} & 0 \\ 0 & e^{-i\tilde{\omega}t/2}\end{pmatrix}\begin{pmatrix} 0 & e^{+i\tilde{\omega}t/2} \\ e^{-i\tilde{\omega}t/2} & 0\end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0\end{pmatrix} = \begin{pmatrix} 0 & \cos\omega t + i\sin\omega t \\ \cos\omega t - i\sin\omega t & 0\end{pmatrix}$$

$$= \cos\tilde{\omega}t\,\sigma_x - i\sin\tilde{\omega}t\,\sigma_y$$

$$U\sigma_y U^\dagger = \sin\tilde{\omega}t\,\sigma_x + \cos\tilde{\omega}t\,\sigma_y$$

$$U\dot{U}^\dagger = -i\frac{\tilde{\omega}}{2}\sigma_z$$

$\pm$: $\cos\omega t\left(\cos\tilde{\omega}t\,\sigma_x - \sin\tilde{\omega}t\,\sigma_y\right) \pm \sin\omega t\left(\sin\tilde{\omega}t\,\sigma_x + \cos\tilde{\omega}t\,\sigma_y\right) =$

$\tilde{\omega} = \omega$: $\begin{cases} + : \cos^2\omega t + \sin^2\omega t\,\sigma_x = \sigma_x \\ - : \cos(2\omega t)\,\sigma_x - \sin(2\omega t)\,\sigma_y \end{cases}$ .. rotating at twice the frequency

→ can usually be neglected

resonance: $\omega = \Omega$. $\tilde{H}_{res} = \frac{\varepsilon}{2}\sigma_x$
rotation about σ_x-axis

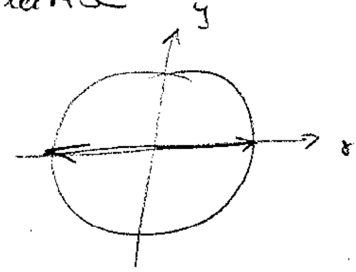
$\Rightarrow \tilde{H} = \frac{1}{2}(\Omega - \omega)\sigma_z + \frac{\varepsilon}{2}\sigma_x$ .. transf. to static Hamiltonian by discarding $\ominus$ term → RWA

Rotating Wave Approximation

$$H = \frac{\Omega}{2}\sigma_z + \varepsilon \cos\omega t\, \sigma_x$$

$\uparrow$ oscillating field in x-direction

$$H = \vec{m}(t)\cdot\vec{\sigma} \qquad m(t) = \begin{pmatrix} \varepsilon\cos\omega t \\ 0 \\ \Omega/2 \end{pmatrix}$$

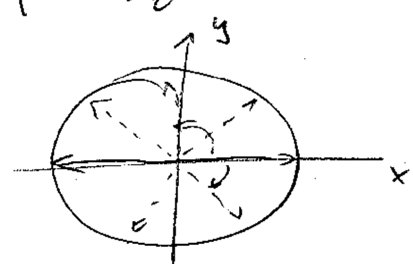


Larmor precession of unperturbed system:

$$|\psi(t)\rangle = e^{-i\frac{\Omega}{2}\sigma_z t}\,|\psi(0)\rangle$$

Decompose $\cos\omega t\,\sigma_x$ term into a component rotating at the Larmor frequency and a counterrotating term

$$m(t) = \frac{1}{2}\begin{pmatrix} \varepsilon\cos\omega t \\ \varepsilon\sin\omega t \\ \Omega/2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \varepsilon\cos\omega t \\ -\varepsilon\sin\omega t \\ \Omega/2 \end{pmatrix}$$



$$H = \frac{\Omega}{2}\sigma_z + \frac{\varepsilon}{2}\Big[\underbrace{\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y}_{\oplus}\Big] + \frac{\varepsilon}{2}\Big[\underbrace{\cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y}_{\ominus}\Big]$$

Rotating frame: $U = e^{i\frac{\tilde{\omega}t}{2}\sigma_z}$

$$|\psi(t)\rangle \rightarrow U|\psi(t)\rangle = U e^{-i\frac{\Omega}{2}\sigma_z t}|\psi(0)\rangle =$$

$$= e^{i\frac{\tilde{\omega}t}{2}\sigma_z} e^{-i\frac{\Omega}{2}\sigma_z t}|\psi(0)\rangle =$$

$$= |\psi(0)\rangle \text{ for } \tilde{\omega} = \Omega$$

$\rightarrow$ stationary state in rotating frame

III/4)

$$\vec{v}\cdot\vec{\sigma} = v_x \cdot \sigma_x + v_y\sigma_y + v_z\sigma_z$$

a)

$$\left.\begin{aligned} v_x\sigma_x &= \begin{pmatrix} 0 & v_x \\ v_x & 0 \end{pmatrix} \\ v_y\sigma_y &= \begin{pmatrix} 0 & -iv_y \\ +iv_y & 0 \end{pmatrix} \\ v_z\sigma_z &= \begin{pmatrix} v_z & 0 \\ 0 & -v_z \end{pmatrix} \end{aligned}\right\} \sigma_v \equiv \vec{v}\cdot\vec{\sigma} = \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix}$$

$$\det \begin{vmatrix} v_z - \lambda & v_x - iv_y \\ v_x + iv_y & -v_z - \lambda \end{vmatrix} = -(v_z - \lambda)(v_z + \lambda) - v_x^2 - v_y^2 =$$

$$= -v_z^2 + \lambda^2 - v_x^2 - v_y^2 =$$

$$= \lambda^2 - 1$$

$$\det |\ \ | = 0 \Rightarrow \lambda^2 = 1 \Rightarrow {}_1\lambda_2 = \pm 1$$

$$\sigma_v |\psi\rangle = {}_1\lambda_2 |\psi\rangle \qquad |\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} v_z - 1 & v_x - iv_y \\ v_x + iv_y & v_z + 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

$$\text{I} \quad (v_z - 1)a + (v_x - iv_y)b = 0$$

$$\text{II} \quad (v_x + iv_y)a + (v_z + 1)b = 0$$

b + c) see Mathematica-File

d) Projection operator: $P_{V+}|\psi^+\rangle = |\psi^+\rangle \qquad P_{v-}|\psi^+\rangle = 0$

$$P_{v-}|\psi^-\rangle = 0 \qquad P_{v-}|\psi^-\rangle = |\psi^-\rangle$$

$$P_+|\psi^+\rangle = \frac{1}{2}(1 + \vec{v}\cdot\vec{\sigma})|\psi^+\rangle = \frac{1}{2}(|\psi^+\rangle + |\psi^+\rangle) = |\psi^+\rangle$$

$$P_+|\psi^-\rangle = \frac{1}{2}(1 + \vec{v}\cdot\vec{\sigma})|\psi^-\rangle = \frac{1}{2}(|\psi^-\rangle - |\psi^-\rangle) = 0$$

z.B. $P_{x+} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad P_{x+}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad P_{x-} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad P_{x-}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Alternative approach for calculating eigenvectors:

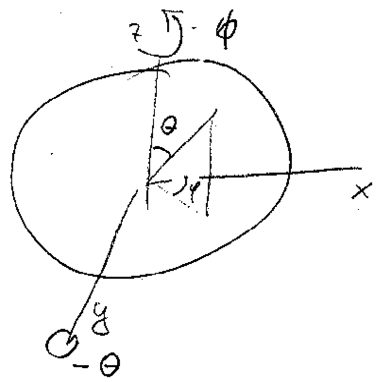
$\vec{v}\cdot\vec{\sigma}$ can be rotated such that $v \to v' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\Rightarrow \quad \vec{v}\vec{\sigma} \to \sigma_z$

$$U^+(\hat{v}\cdot\vec{\sigma})\,U = \sigma_z$$

U.. rotations about z-axis and y-axis (Euler rotations)

$U = U_y(-\theta) \cdot U_z(-\varphi)$



$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ eigenvalues to σ_z

$$\Rightarrow \quad \sigma_z\,|\pm\rangle = \pm\,|\pm\rangle$$

$$\sigma_z \underbrace{U^+ U}_{\mathbb{1}}\,|\pm\rangle = \pm\,|\pm\rangle$$

$$\underbrace{U\sigma_z U^+}_{\vec{v}\cdot\vec{\sigma}}\;\underbrace{U|\pm\rangle}_{\text{eigenvector to } \vec{v}\cdot\vec{\sigma}} = \pm\,U|\pm\rangle$$

$\cancel{U = U_y(\theta)\,U_z(\varphi)}$