QSIT Course 2009 - Problem Sheet 2

Peter Leek / Stefan Filipp

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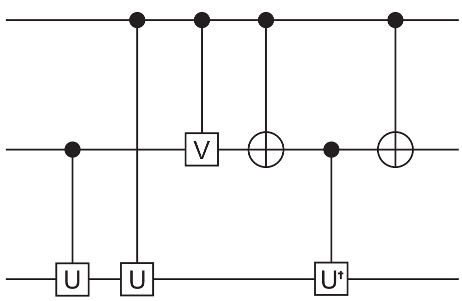
1. (a) In classical logic, the gates NOT, OR, AND together are universal for computation, as is the NAND gate alone. Why can these not be translated directly into quantum gates?

   (b) If we additionally require classical computation to be reversible, it turns out that a 3-bit gate is required for universal computation - the Toffoli gate:

$$U_{\text{Toffoli}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

   Show that it is possible to reproduce this gate with the following quantum circuit of 2 qubit gates:



   where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$.

2. The Pauli operators σ_x, σ_y and σ_z correspond to rotations of a qubit state on the Bloch sphere around the x, y and z axes respectively. We can define a more general operator $\mathbf{v}.\sigma \equiv \mathbf{v_x}\sigma_\mathbf{x} + \mathbf{v_y}\sigma_\mathbf{y} + \mathbf{v_z}\sigma_\mathbf{z}$, where $\mathbf{v}$ is a 3D unit vector.

   (a) Show that the eigenvalues of this operator are ± 1.

   (b) What are the corresponding eigenvectors?

   (c) Show that the eigenvectors correspond to unit vectors pointing in the direction $\pm\mathbf{v}$ on the Bloch sphere.

   (d) Show that the projection operators onto the eigenspace of $\mathbf{v}.\sigma$ are $P_\pm = (I \pm \mathbf{v}.\sigma)/2$, corresponding for example to an z-axis projection operator $Z = (1 + \sigma_z)/2$.