QSIT 2010 - Questions 4

27. Oktober 2010

1. **Bloch vector**
   Check, if the following density matrices ($\rho = \frac{1}{2}\left(\mathrm{id} + \vec{r}\cdot\vec{\sigma}\right)$) are physical, calculate their Bloch vector $\vec{r}$ and plot them on the Bloch sphere.

   (a) $\rho = \begin{pmatrix} 0.65 & 0 \\ 0 & 0.35 \end{pmatrix}$

   (b) $\rho = \begin{pmatrix} 0.65 & 0 \\ 0 & 0.35 \end{pmatrix}$

   (c) $\rho = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$

2. **Mixed state vs. pure state**
   A source emits spin-1/2 particles either in the pure state

   $$|\psi\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$$

   or in the mixed state

   $$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

   (a) The spin polarization of the particles is then analysed by performing a projective measurement along the axis $n = (\sin\phi, 0, \cos\phi)^T$) in the $x-z$ plane. How can you distinguish the pure from the mixed state in this measurement?

   (b) What is observed for a totally mixed state?

   (c) For a full state tomography, three different measurements along different axes have to be performed. Calculate the expectation values for measurements along the (positive) $x-$, $y-$ and $z-$axis for the mixed state ρ and reconstruct the state from these measurement outcomes.

3. **Thermal state of a harmonic oscillator**
The Hamiltonian of a harmonic oscillator is given by

$$\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2})$$

with the number operator $\hat{n}$. The eigenstates, which fulfill the Schrödinger equation $\hat{H}|n\rangle = \hbar\omega(n+1/2)|n\rangle$, are denoted by $|0\rangle$, $|1\rangle$ $\ldots$, $|n\rangle$ and correspond to states with n excitation quanta in the system.

In thermal equilibrium with a heat bath at temperature T the probability p_n that the harmonic oscillator is excited to the nth state is given by the Boltzmann distribution

$$p_n = \frac{\exp[-E_n/(k_B T)]}{\sum_n \exp[-E_n/(k_B T)]} = \left(1 - \exp\left[-\frac{\hbar\omega}{k_B T}\right]\right) \exp\left[-\frac{n\hbar\omega}{k_B T}\right].$$

Find the density matrix which describes the equilibrium state of the harmonic oscillator.