Dynamics of a Quantum System:

QM postulate: The time evolution of a state $|\psi\rangle$ of a closed quantum system is described by the Schrödinger equation

$it \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$

H = Z E IEZCEI

where H is the hermitian operator known as the Hamiltonian describing the closed system.

a closed quantum system does not interact with any other system

general solution:

$$|\Psi(t)\rangle = e \times p \left[\frac{-iHt}{h}\right] |\Psi(0)\rangle$$

the Hamiltonian:

- H is hermitian and has a spectral decomposition

example:

- and eigenvectors E>
- smallest value of € is the ground state energy with the eigenstate | E>

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$$E = 113$$

$$E = 113$$

$$E = 103$$

e.g. electron spín ín a field:



on the Bloch sphere:



$$H = -\frac{t_{1}\omega}{2} Z$$

$$H = -\frac{t_{1}\omega}{2} (10)(0(-1))(1)$$

$$|\Psi(0)\rangle = 10\rangle \longrightarrow |\Psi(1)\rangle = e^{\frac{i\omega}{2}t} |0\rangle$$

$$|\Psi(0)\rangle = 11\rangle \longrightarrow |\Psi(1)\rangle = e^{-\frac{i\omega}{2}t} |1\rangle$$

$$|\Psi(0)\rangle = \frac{1}{52} (10) + |1\rangle)$$

$$= \frac{1}{52} e^{\frac{i\omega}{2}t} (10) + e^{-i\omega t} |1\rangle$$

$$|\Psi\rangle = e^{\frac{i\omega}{2}t} (\cos \frac{2}{2} |0\rangle + e^{\frac{i\omega}{2}t} \sin \frac{2}{2} |1\rangle$$

$$= 1\rangle \quad \Theta = \frac{t_{1}}{2}, \quad \Psi = -\omega t$$

this is a rotation around the equator with Larmor precession frequency $\boldsymbol{\omega}$

Rotation operators:

when exponentiated the Pauli matrices give rise to rotation matrices around the three orthogonal axis in 3-dimensional space.

$$R_{x}(\theta) = e^{-i\theta X/z} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_{y}(\theta) = e^{-i\theta Y/z} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_{z}(\theta) = e^{-i\theta Z/z} - \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Z = \begin{pmatrix} e^{-i\theta X} & 0 \\ 0 & e^{-i\theta X} \end{pmatrix}$$
If the Pauli matrices X Y or Z are present in the Hamiltonian of a system they will give rise to rotations of the qubit state vector around the respective axis.
Exercise: convince yourself that the operators R_{xyz} do perform rotations on the qubit state written in the Bloch sphere representation.
QSTUD 2 Page 3
Control of Single Qubit States
by resonant irradiation: $E \int_{-1}^{-1} \int_{-1}^{-1$

qubit Hamiltonian with ac-drive:



L, (05(0t)

 $\Delta \vec{e} = \hbar \omega_q$



ac-fields applied along the x and y components of the qubit state



i.e. irradiating the qubit with an ac-field with controlled amplitude and phase all realize arbitrary single qubit rotations.



in fact such a pulse of chosen length and phase can prepare any single qubit state, i.e. any point on the Bloch sphere can be reached

Quantum Measurement

One way to determine the state of a qubit is to measure the projection of its state vector along a given axis, say the z-axis.

On the Bloch sphere this corresponds to the following operation:

z y



After a projective measurement is completed the qubit will be in either one of its computational basis states.

In a repeated measurement the projected state will be measured with certainty.

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QM postulate: quantum measurement is described by a set of operators $\{M_m\}$ acting on the state space of the system. The probability p of a measurement result m occurring when the state ψ is measured is

the state of the system after the measurement is

$$|\psi'\rangle = \frac{M_{M}|\psi\rangle}{\sqrt{p(m)}}$$

completeness: the sum over all measurement outcomes has to be unity

$$I = \sum_{nm} p(nm) = \sum_{nm} \langle \Psi | M_{nm}^{\dagger} M_{nm} | \Psi \rangle$$

example: projective measurement of a qubit in state ψ in its computational basis

 $|47 = \propto 107 + \beta 117$

measurement operators:

$$|M_0\rangle = 10\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; M_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

measurement probabilities:

$$p(o) = (4|M_0^{\dagger}M_0|4) = \alpha^* \alpha (0|0) = |\alpha|^2$$

 $p(i) = (4|M_1^{\dagger}M_1|4) = \beta^* \beta (111) = |\beta|^2$

state after measurement:



measuring the state again after a first measurement yields the same state as the initial measurement with unit probability

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information content in a single qubit:



- infinite number of qubit states
- but single measurement reveals only 0 or 1 with probabilities (alor (B)²
- measurement will collapse state vector on basis state
- to determine x and b an infinite number of measurements has to be made

But, if not measured qubit contains 'hidden' information about \propto and β .

A Few Physical Realizations of Qubits

nuclear spins in molecules:

energy scales: 1 GHz = 50 mk I Gitz = 4 mel

nuclear magnetic moment in external magnetic field



106 × DE for muclei QSIT07.2 Page 12

(http://heart-c704.uibk.ac.at/)





polarization states of photons:

are used in quantum

cryptography and for

the one-way quantum

quantum

computer

Two Qubits:

6:41

С

O

1

communication

qubit states corresponding to different polarizations of a single photon (in the visible frequency range)

Photon Polarization



 2^n different states (here n=2)

6:62

0

1 D

but only one is realized at any given time

- 2^n basis states (n=2)
- can be realized simultaneously
- quantum parallelísm

2ⁿ complex coefficients describe quantum state

145 = a 1005 + a (101) + a (10) + a (11) $\sum_{i:1} |\alpha_{i:1}|^2 = 1$

normalization condition

Composíte quantum systems

QM postulate: The state space of a composite systems is the tensor product of the state spaces of the component physical systems. If the component systems have states ψ_i the composite system state is

$(\Psi) = |\Psi_{z}\rangle \otimes |\Psi_{z}\rangle \otimes ... \otimes |\Psi_{m}\rangle$

This is a product state of the individual systems.

example:

$$|\Psi_1\rangle = |\Psi_1|0\rangle + \beta_1|1\rangle$$

$$|\Psi_1\rangle = |\Psi_2|0\rangle + \beta_2|1\rangle$$

 $| \Psi_2 \rangle^{=} \propto_{2} | \psi_1 \rangle + \#_2 | \psi_1 \rangle$ -0 $| \Psi_1 \rangle = | \Psi_1 \rangle \otimes | \Psi_2 \rangle = | \Psi_1 \Psi_2 \rangle$ = $\varphi_1 \varphi_2 | 007 + \varphi_1 \beta_2 | 017 + \beta_1 \varphi_2 | 107 + \beta_1 \beta_2 | 117)$

exercíse: Write down the state vector (matrix representation) of two qubits, i.e. the tensor product, in the computational basis. Write down the basis vectors of the composite system.

there is no generalization of Bloch sphere picture to many qubits

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Information content in multiple qubits

- 2ⁿ complex coefficients describe state of a composite quantum system with n qubits!
- Imagine to have 500 qubits, then 2⁵⁰⁰ complex coefficients describe their state.
- How to store this state. 2⁵⁰⁰ is larger than the number of atoms in the universe. It is impossible in classical bits. This is also why it is hard to simulate quantum systems on classical computers.
- A quantum computer would be much more efficient than a classical computer at simulating quantum systems.
- Make use of the information that can be stored in qubits for quantum information processing!

Operators on composite systems:

Let A and B be operators on the component systems described by state vectors $|a\rangle$ and $|b\rangle$. Then the operator acting on the composite system is written as

 $A \otimes B(1a) \otimes 165) = A 1a > \otimes B 16$

tensor product in matrix representation (example for 2D Hilbert spaces):

$$A \otimes B = \begin{pmatrix} A_{11} & B & A_{12} & B \\ A_{21} & B & A_{22} & B \end{pmatrix}$$

$$|a\rangle \otimes |b\rangle = \begin{pmatrix} a_{1} & |b\rangle \\ a_{2} & |b\rangle \end{pmatrix} = \begin{pmatrix} a_{1} & b_{1} \\ a_{1} & b_{2} \\ a_{2} & b_{1} \\ a_{2} & b_{2} \end{pmatrix}$$

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Entanglement:

Definition: An **entangled state** of a composite system is a state that cannot be written as a product state of the component systems.

example: an entangled 2-qubit state (one of the Bell states)

$$|47 = \frac{1}{\sqrt{2}} (1007 + (117))$$

What is special about this state? Try to write it as a product state!

$$\begin{aligned} |\Psi_{1}\rangle &= \alpha_{1}(0) + \beta_{1}(1); \quad (\Psi_{2}) = \alpha_{2}(0) + \beta_{2}(1) \\ |\Psi_{1}\Psi_{2}\rangle &= \alpha_{1}\alpha_{2}(00) + \alpha_{1}\beta_{2}(01) + \beta_{1}\alpha_{2}(10) + \beta_{1}\beta_{2}(11) \\ |\Psi_{1}\Psi_{2}\rangle &= \alpha_{1}\alpha_{2}(0) + \alpha_{1}\beta_{2}(01) + \beta_{1}\alpha_{2}(10) + \beta_{1}\beta_{2}(10) \\ |\Psi_{1}\Psi_{2}\rangle &= \alpha_{1}\alpha_{2} + \frac{1}{12} \int \beta_{1}\beta_{2} = \frac{1}{12} \int \beta_{2}\beta_{2} + 0 \\ &= A\alpha_{2}\beta_{1}\phi_{1}^{2} \\ &= A\alpha_{2}\beta_{1}\phi_{1}^{2} \end{aligned}$$

It is not possible! This state is special, it is entangled!

Measurement of single qubits in an entangled state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(100\right) + |11\rangle$$

measurement of first qubit:

$$\rho_1(0) = \langle \Psi | (N_0 \otimes I)^+ (N_0 \otimes I) | \Psi \rangle = \frac{1}{\sqrt{2}} \langle 00| \frac{1}{\sqrt{2}} | 00 \rangle = \frac{1}{2}$$

post measurement state:

$$|\psi'\rangle = \frac{(M_{2}\otimes I)|\psi\rangle}{\sqrt{\rho_{1}(0)}} = \frac{\frac{1}{\sqrt{2}}|00\rangle}{\frac{1}{\sqrt{2}}} = (00)$$

measurement of qubit two will then result with certainty in the same result:

$$P_{2}(o) = \langle \Psi^{\dagger} | (I \otimes M_{o})^{\dagger} (I \otimes M_{o}) | \Psi^{\dagger} \rangle = 1$$

The two measurement results are **correlated**! Correlations in quantum systems can be stronger than correlations in classical systems. This can be generally proven using the **Bell inequalities** which will be discussed later. Make use of such correlations as a **resource** for information processing, for example in **super dense coding** and **teleportation**.



- CNOT is reversible (unitary)

universality of controlled NOT:

Any multí qubit logic gate can be composed of CNOT gates and single qubit gates X,Y,Z.

application of CNOT: generation of entangled states (Bell states):



exercise: Write down the unitary matrix representations of the CNOT in the computational basis with qubit 1 being the control qubit. Write down the matrix in the same basis with qubit 2 being the control bit.



BUT this does not realize a CNOT gate yet. Additionally, single qubit operations on each of the qubits are required to realize a CNOT gate.

CNOT realization with the Ising-type interaction:

CNOT - unitary:

 $C_{NTT} = e^{-i\frac{3T}{4}} R_{X_2} \left[\frac{3T}{2} \right] C \left(\frac{3T}{2} \right) R_{Z_2} \left(\frac{T}{2} \right) R_{X_2} \left(\frac{T}{2} \right) R_{Z_1} \left(\frac{T}{2} \right) C \left(\frac{3T}{2} \right)$

círcuít representation:



Any physical two-qubit interaction that can produce entanglement can be turned into a universal two-qubit gate (such as the CNOT gate) when it is augmented by arbitrary single qubit operations. [Bremner et al., PRL **89**, 247902 (2002)]

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Quantum Teleportation:

Task: Alice wants to transfer an unknown quantum state ψ to Bob only using one entangled pair of qubits and classical information as a resource.

note:

- Alice does not know the state to be transmitted

- Even if she knew it the classical amount of information that she would need to send would be infinite.

The teleportation circuit:



original article:

Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters Phys. Rev. Lett. **70**, 1895 (1993) [PROLA Link] How does it work?

$(14) \otimes \frac{1}{\sqrt{2}} (1007 + 111)) = \frac{1}{\sqrt{2}} (\propto 10007 + \alpha 1011 + \beta 11007 + \beta 1111)$

CNOT between qubit to be teleported and one bit of the entangled pair:

(2)
$$\xrightarrow{C \operatorname{Not}_{12}} \frac{1}{\sqrt{2}} (\alpha | opp) + \alpha | o(1) + \beta | 100 + \beta | 101 \rangle$$

Hadamard on qubit to be teleported:

(3)
$$\frac{H_1}{2} = \frac{1}{2} \left[(100) (\alpha 10) + \beta 110 + 110) (\alpha 10) - \beta 11) \right] + 101 (\alpha (1) + \beta (0)) + 111 (\alpha (1) - \beta 10) \right]$$

measurement of qubit 1 and 2, classical information transfer and single bit manipulation on target qubit 3:

$$(\Psi) \xrightarrow{\mu_{1} \otimes \mu_{2}} P_{00} = \frac{1}{4} ; (\Psi_{3}^{2}) = \alpha |0\rangle + \beta (1) \xrightarrow{\Sigma} |\Psi\rangle$$

$$P_{10} = \frac{1}{4} ; |\Psi_{3}^{2}\rangle = \alpha |0\rangle - \beta (1) \xrightarrow{\Sigma} |\Psi\rangle$$

$$P_{01} = \frac{1}{4} ; |\Psi_{3}^{2}\rangle = \alpha |1\rangle + \beta |0\rangle \xrightarrow{X} |\Psi\rangle$$

$$P_{11} = \frac{1}{4} ; |\Psi_{3}^{2}\rangle = \alpha |1\rangle - \beta |0\rangle \xrightarrow{X \ge} |\Psi\rangle$$

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(One) Experimental Realization of Teleportation using Photon Polarization:





- parametric down conversion (PDC)
 source of entangled photons
- qubits are polarization encoded

Experimental quantum teleportation Dik Bouwmeester, Jian-Wei Pan, Klaus Mattle, Manfred Eibl, Harald Weinfurter, Anton Zeilinger Nature 390, 575 - 579 (11 Dec 1997) Article <u>Abstract</u> | <u>Full Text</u> | <u>PDF</u> | <u>Rights and permissions</u> | <u>Save this link</u> Experimental Implementation

start with states

 $|\psi_{1}\rangle = \propto |H\rangle + \beta |U\rangle$ $|\psi_{23}\rangle = \frac{1}{\sqrt{2}} \left(|HV\rangle - |VH\rangle\right)$

combine photon to be teleported (1) and one photon of entangled pair (2) on a 50/50 beam splitter (BS) and measure (at Alice) resulting state in Bell basis.

analyze resulting teleported state of photon (3) using polarizing beam splitters (PBS) single photon detectors



- polarízíng beam splítters (PBS) as detectors of teleported states

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teleportation papers for you to present:

Experimental Realization of Teleporting an Unknown Pure Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu Phys. Rev. Lett. **80**, 1121 (1998) [PROLA Link]

Unconditional Quantum Teleportation

A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik Science 23 October 1998 282: 706-709 [DOI: 10.1126/science.282.5389.706] (in Research Articles) Abstract » Full Text » PDF »

Complete quantum teleportation using nuclear magnetic resonance

M. A. Nielsen, E. Knill, R. Laflamme Nature 396, 52 - 55 (05 Nov 1998) Letters to Editor <u>Abstract | Full Text | PDF | Rights and permissions</u> | <u>Save this link</u>

Deterministic quantum teleportation of atomic qubits M. D. Barrett, J. Chiaverini, T. Schaetz, J. Britton, W. M. Itano, J. D. Jost, E. Knill, C. Langer, D. Leibfried, R. Ozeri, D. J. Wineland Nature 429, 737 - 739 (17 Jun 2004) Letters to Editor <u>Abstract | Full Text | PDF | Rights and permissions | Save this link</u>

Deterministic quantum teleportation with atoms

M. Riebe, H. HĤffner, C. F. Roos, W. HĤnsel, J. Benhelm, G. P. T. Lancaster, T. W. KĶrber, C. Becher, F. Schmidt-Kaler, D. F. V. James, R. Blatt Nature 429, 734 - 737 (17 Jun 2004) Letters to Editor <u>Abstract | Full Text | PDF | Rights and permissions | Save this link</u>

Quantum teleportation between light and matter Jacob F. Sherson, Hanna Krauter, Rasmus K. Olsson, Brian Julsgaard, Klemens Hammerer, Ignacio Cirac, Eugene S. Polzik Nature 443, 557 - 560 (05 Oct 2006) Letters to Editor Full Text | PDF | Rights and permissions | Save this link