1.11. Quantum Teleportation

Dienstag, 06. März 2012
18:57

(Bennett et al 1993)

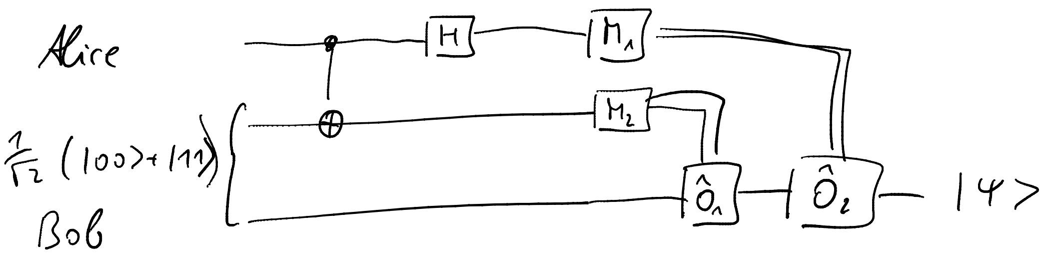
Task: Transfer unknown quantum state $|\psi\rangle$ from Alice to Bob at a distance

What are the options of Alice?
- transport state to Bob
- measure state, send classical information
- broadcast quantum information

Why is this difficult in quantum mechanics?
- quantum states cannot be measured reliably
- quantum states are fragile
- no-cloning theorem → no broadcasting of quantum information

Resources: entangled pair of qubits & classical communication

Circuit:



Steps:
1. input (entangled pair)
2. CNOT
3. Hadamard
4. measurement
5. conditional operations
6. output

Note:
- Alice has no information about $|\psi\rangle$ (and cannot obtain it)
- state is always fully transferred

Teleportation Protocol

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① initial state: $(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) =$

$= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

② $CNOT_{12}$: $\Rightarrow \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

③ Hadamard$_1$: $\Rightarrow \frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle -$

$- \beta|110\rangle + \beta|010\rangle - \beta|101\rangle + \beta|001\rangle)$

$$= \frac{1}{2}\{|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\}$$

④ measurement of qubits 1,2

$M_1 \otimes M_2 \otimes \mathbb{1}$

$P_{00} = P_{10} = P_{01} = P_{11} = \frac{1}{4}$

⑤ conditional manipulations on post measurement state $|\psi'\rangle$

$$\left.\begin{array}{ll} |00\rangle: & \mathbb{1}|\psi'\rangle \\ |10\rangle: & \hat{z}|\psi'\rangle \\ |01\rangle: & \hat{x}|\psi'\rangle \\ |11\rangle: & \hat{z}\hat{x}|\psi'\rangle \end{array}\right\} = \alpha|0\rangle + \beta|1\rangle$$

⇒ requires transfer of two bits of classical info to Bob to perform local operations that recover original state

Applications: ∘) Quantum error correction
∘) Quantum gates
∘) Quantum repeaters

Experiments: D. Bouwmeester et al Nature 390, 575 (1997) (first with photons)

hallmark experiment for quantum information processing

tested with nuclear spins, ions, partly with superconducting qubits

1.12 Mixed States

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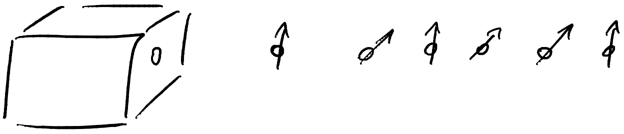
How can we describe a statistical mixture of pure states, i.e. the state of a quantum system of which we have only partial information?



e.g.

$\lvert\psi\rangle$	prob
$\uparrow$	0.5
$\nearrow$	0.5

pure states $|\psi_i\rangle = \cos\frac{\theta_i}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle$ $\{\theta_i = 0, \delta\}$ with probabilities $\{p_1, p_2\}$.

$\Rightarrow$ $|\tilde{\psi}\rangle = p_1|\psi_1\rangle + p_2|\psi_2\rangle$? Is this the proper description ?

Does not describe statistical mixture!
Main 'problem': coherence between basis states leads to interference!

Density matrix:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \qquad \sum p_i = 1$$

$$\Rightarrow \rho_{ex} = 0.5\,|\uparrow\rangle\langle\uparrow| + 0.5\,|\nearrow\rangle\langle\nearrow|$$

$$|\uparrow\rangle\langle\uparrow| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$|\nearrow\rangle\langle\nearrow| = \begin{pmatrix}\cos\frac{\delta}{2}\\ \sin\frac{\delta}{2}\end{pmatrix}\left(\cos\frac{\delta}{2}, \sin\frac{\delta}{2}\right) = \begin{pmatrix}\cos^2\delta/2 & \cos\frac{\delta}{2}\sin\frac{\delta}{2}\\ \cos\frac{\delta}{2}\sin\frac{\delta}{2} & \sin^2\frac{\delta}{2}\end{pmatrix}$$

$$\rho = \frac{1}{2}\begin{pmatrix}1+\cos^2\delta/2 & \cos\frac{\delta}{2}\sin\frac{\delta}{2}\\ \cos\frac{\delta}{2}\sin\frac{\delta}{2} & 1+\sin^2\frac{\delta}{2}\end{pmatrix}$$

General parameterization & properties

$$\rho = \begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}$$

- ρ is hermitian: $\rho^\dagger = \rho$ (real eigenvalues $\hat{=}$ real valued probabilities)
- $\mathrm{Tr}[\rho] = 1$ $\quad (\sum p_i = 1)$
- positive semidefinite $\hat{=}$ all eigenvalues ≥ 0 (no negative probabilities)

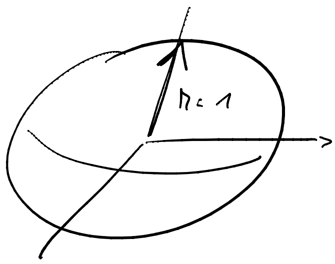
e.g. for $J = \pi$: $\rho = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$p_\uparrow = p_\downarrow = 0.5$$

<u>Bloch ball representation</u>: $\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})$ $\quad |\vec{r}| \leq 1$

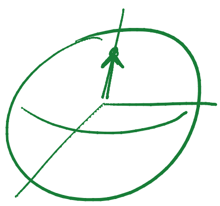
$\vec{r}$... vector inside Bloch sphere

e.g. - $\vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\hat{=}$ $|1\rangle\langle 1|$

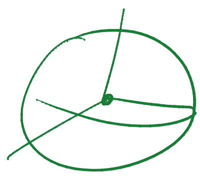


$\Rightarrow$ pure states have $\underline{|\vec{r}| = 1}$!

- $\vec{r} = \begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix}$? $\quad \rho = \frac{1}{2}(\mathbb{1} + 0.5\,\sigma_z) = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$



- $\vec{r}$ for $p_1 = p_2 = 0.5$? $\quad \rho = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$



$\rightarrow$ totally mixed state, no information about state of system

Measurement

operator $\hat{M}$: $\langle \hat{M} \rangle = \mathrm{Tr}[\hat{M}\hat{\rho}]$ ... average value of $\hat{M}$

$$\mathrm{Tr}[\hat{M}\rho] = \sum_k \langle k| M\rho |k\rangle \quad \text{.. sum over basis states}$$

$$= \sum_{k,l} \langle k| M |l\rangle\langle l| \rho |k\rangle =$$

$$= \sum_{k,l} \langle k| M |l\rangle \sum_m p_m \langle \underbrace{l|m}_{\delta_{lm}}\rangle\langle \underbrace{m|k}_{\delta_{mk}}\rangle =$$

$$= \sum_m \langle m| M |m\rangle p_m$$

↳ expectation value for pure state $|m\rangle$

$(p_1 = 1, p_0 = 0 : \mathrm{Tr}[M\rho] = \langle 1|M|1\rangle \hat{=}$
$\langle \psi|M|\psi\rangle = \langle M\rangle_\psi$

e.g.: determine coeffs of $\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})$

$$\mathrm{Tr}[\rho\,\sigma_{x,y,z}] = r_x, r_y, r_z$$

Unitary evolution :

$$\rho(t) = U\rho(0)U^+$$

$$\left(\text{pure state: } |\psi(t)\rangle\langle\psi(t)| = U|\psi(0)\rangle\langle\psi(0)|U^+\right)$$

1.13 Bloch Equations, T1 & T2 times

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Unitary Evolution: $H = -\mu \cdot \vec{\sigma} \cdot \vec{m}(t)$ $\quad \vec{m}(t) = \begin{pmatrix} m_x(t) \\ m_y(t) \\ m_z(t) \end{pmatrix}$

(Liouville-) von Neumann equation:

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] = -\frac{i}{\hbar}[H \cdot \rho - \rho \cdot H] \qquad \rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma})$$

$$\dot{\rho} = \frac{i\mu}{\hbar}\left[\vec{\sigma} \cdot \vec{m} \cdot \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) - \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma})\,\vec{\sigma} \cdot \vec{m}\right] =$$

$$= \frac{i\mu}{2\hbar}\left[(\vec{m} \cdot \vec{\sigma}) \cdot (\vec{r} \cdot \vec{\sigma}) - (\vec{r} \cdot \vec{\sigma})(\vec{\sigma} \cdot \vec{m})\right]$$

using $(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = (\vec{a} \cdot \vec{b})\mathbb{1} + i(\vec{a} \times \vec{b})\vec{\sigma}$

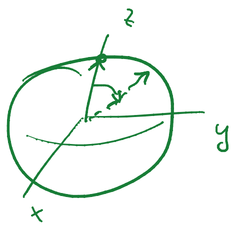
$$= \frac{i\mu}{\hbar}(-i\,\vec{r} \times \vec{m})\,\vec{\sigma}$$

$$= \frac{\mu}{\hbar}(\vec{r} \times \vec{m}) \cdot \vec{\sigma}$$

in terms of Bloch vector components:

$$\dot{\vec{r}} = \mathrm{Tr}[\dot{\rho}\,\vec{\sigma}] = \overset{\gamma}{\frac{\mu}{\hbar}}(\vec{r} \times \vec{m}) \quad \ldots \text{ 'Bloch equations'}$$

e.g.: $\vec{r}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad m(t) = m_x$



$$\dot{r}_x = \gamma(r_y m_z - r_z m_y) = 0$$

$$\left.\begin{aligned} \dot{r}_y &= \gamma(r_z m_x - r_x m_z) = \gamma r_z m_x \\ \dot{r}_z &= \gamma(r_x m_y - r_y m_x) = -\gamma r_y m_x \end{aligned}\right\} \text{ coupled diff. eq.}$$

Can one obtain a mixed state ($|\vec{r}| < 1$) out of a pure state ($|\vec{r}| = 1$)?

⇒ introduce energy relaxation and dephasing!

Dissipative evolution:

$$\dot{r}_x = \gamma(\vec{r}\times\vec{m})_x - \frac{r_x}{T_2}$$

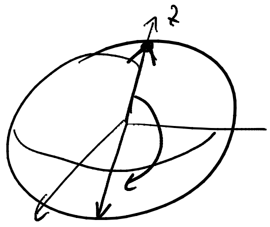
$$\dot{r}_y = \gamma(\vec{r}\times\vec{m})_y - \frac{r_y}{T_2}$$

$$\dot{r}_z = \gamma(\vec{r}\times\vec{m})_z - \frac{r_z+1}{T_1}$$

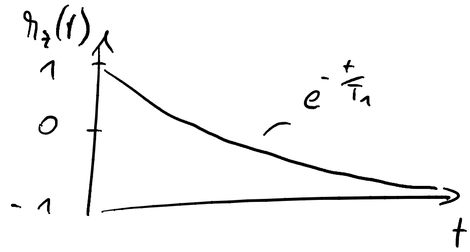
Energy relaxation: $\vec{m}=\vec{0}$, $T_2=0$, $\boxed{T_1\neq 0}$

$$\dot{r}_z = -(r_z+1)/T_1$$

$$r_z(t) = -1+2e^{-t/T_1} \quad \text{for } r_z(0)=1$$

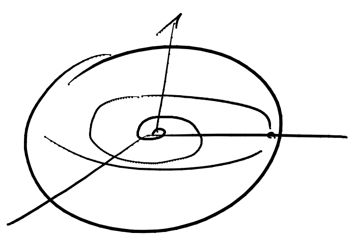


$\Rightarrow T_1$ .. longitudinal relaxation time, energy relaxation



Dephasing: $\vec{m}=m_z$, $T_1=0$, $\boxed{T_2\neq 0}$

$$\left.\begin{aligned}\dot{r}_y &= -\underbrace{\gamma r_x m_z}_{\text{unitary}} - \underbrace{\frac{r_y}{T_2}}_{\text{dissipative}}\\ \dot{r}_x &= \gamma r_y m_z - \frac{r_x}{T_2}\end{aligned}\right\} \; r_x(t), r_y(t) \propto e^{-t/T_2}$$



T_2: transverse relaxation time, dephasing