Qubit manipulation and rotating wave approximation

S. Filipp

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1 Two-Level approximation

In the two-level (qubit) approximation we only take the lowest two energy eigenstates into account, i. e. we truncate the sum in in the Cooper-pair box Hamiltonian at $N = 1$,

$$H_2 = E_C N_g^2 |0\rangle\langle 0| + E_C(1-N_g)^2 |1\rangle\langle 1| - \frac{E_J}{2} \underbrace{(|0\rangle\langle 1| + |1\rangle\langle 0|)}_{\bar{\sigma}_x} \tag{1}$$

$$= -\frac{E_{el}}{2} \underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|_{\bar{\sigma}_z}} - \frac{E_J}{2}\bar{\sigma}_x \tag{2}$$

$$= -\frac{E_{el}}{2}\bar{\sigma}_z - \frac{E_J}{2}\bar{\sigma}_x \tag{3}$$

where we have shifted the zero energy level by $-E_C(1-2N_g+2N_g^2)$. The electric energy is then given by $E_{el} = E_C(1-2N_g)$. In matrix form the Hamiltonian reads

$$H_2 = \frac{1}{2}\begin{pmatrix} -E_{el} & -E_J \\ -E_J & E_{el} \end{pmatrix} \tag{4}$$

To transform the Hamiltonian into the eigenbasis we can apply the rotation $UHU^\dagger$ with

$$U = e^{i\frac{\theta}{2}\sigma_y} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{5}$$

with $\theta = \arctan(E_J/E_{el})$, which corresponds to a rotation about the y-axis. That this operation diagonalises the Hamiltonian can easily be seen by writing $H_2 = (\omega_q/2)\vec{n}\cdot\vec{\sigma}$ where $\vec{n} = (\sin\theta', 0, \cos\theta')^T$ with $\theta' = -\arctan(E_J/E_{el})$ as the angle to the z-axis (also called the mixing angle) and $\omega_q = \sqrt{E_J^2 + E_{el}^2}$. A rotation by $-\theta'$ rotates the Hamiltonian to 'point in the z-direction',

$$H_2^{diag} = \frac{\omega_q}{2}\sigma_z.$$

In most of the experiments the qubit is operated at the 'charge sweet spot' at $N_g = 1/2$ at which the curvature of the energy levels as a function of the gate charge is flat. In this case $\theta = \pi/2$ and the Pauli operators transform in a simple manner,

$$\sigma_x \to -\sigma_z \tag{6}$$

$$\sigma_z \to \sigma_x \tag{7}$$

2 Single qubit control

We can write the polarisation charge $N_g = N_g^0 + N_g(t)$ at the gate capacitor as a sum of a dc-bias N_g^0 plus a time-dependent part $N_g(t) = \eta \cos(\omega t + \phi)$

$$H = H_r - \frac{E_C(1 - 2(N_g^0 + \eta \cos(\omega t + \phi)))}{2} \bar{\sigma}_z - \frac{E_J}{2} \bar{\sigma}_x.$$

Working at the charge sweet spot with $N_g^0 = 1/2$ we obtain

$$H = H_r - \underbrace{E_C \eta}_{\Omega} \cos(\omega t + \phi) \bar{\sigma}_z - \frac{E_J}{2} \bar{\sigma}_x.$$

We then rotate the basis to obtain

$$H_{CPB} = \frac{\omega_q}{2} \sigma_z + \Omega \cos(\omega t + \phi) \sigma_x = H_r + \vec{m}(t) \cdot \vec{\sigma} \tag{8}$$

where the qubit drive strength is $\Omega \equiv E_C \eta$, equivalent to a spin-1/2 particle in the time-dependent magnetic field

$$\vec{m}(t) = \begin{pmatrix} \Omega \cos(\omega t + \phi) \\ 0 \\ \omega_q/2 \end{pmatrix}$$

This Hamiltonian describes a driven two level system. Note that it does not take the filtering of the drive signal from the resonator's Lorentzian line shape into account: If the qubit is driven through the resonator with a signal of strength ϵ the corresponding drive strength is $\Omega = \epsilon\kappa/\Delta$, with the linewidth κ of the resonator and the detuning of the qubit from the resonator $\Delta = \omega_r - \omega_q$.

2.1 Rotating Wave approximation

The Hamilonian in Eq. (8) is explicitly time-dependent and eludes itself from an analytical solution. We can, however, transform the Hamiltonian into the rotating frame, which rotates at the frequency of the drive ω. To see what this means we can decompose the oscillating field pointing along the x-axis into a component rotating clockwise and a component rotating counter-clockwise in the $x - y$ plane. In simple vector notation this corresponds to writing

$$\cos(\omega t + \phi) e_x = \frac{1}{2} \left(\cos(\omega t + \phi) e_x + \sin(\omega t + \phi) e_y \right) + \frac{1}{2} \left(\cos(\omega t + \phi) e_x - \sin(\omega t + \phi) e_y \right). \tag{9}$$

In terms of the magnetic field $m(t)$ this corresponds to

$$\vec{m}(t) = \begin{pmatrix} \Omega \cos(\omega t + \phi) \\ 0 \\ \frac{\omega_q}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Omega \cos(\omega t + \phi) \\ \Omega \sin(\omega t + \phi) \\ \frac{\omega_q}{2} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \Omega \cos(\omega t + \phi) \\ -\Omega \sin(\omega t + \phi) \\ \frac{\omega_q}{4} \end{pmatrix} \tag{10}$$

Without the drive the qubit will freely evolve due to the presence of the σ_z term, i.e. the presence of an effective magnetic field. Solving Schrödinger equations leads to a time evolution of

$$|\psi(t)\rangle = e^{-i \frac{\omega_q}{2} \sigma_z t} |\psi(0)\rangle, \tag{11}$$

the ground state will pick up a positive phase, the excited state a negative phase, and a superposition state $(|0\rangle + |1\rangle)/\sqrt{2}$ will pick up a relative phase difference between ground and excited state of $\omega_q t$ which corresponds to a counter-clockwise Larmor (right-hand rule) pressesion about the magnetic field ($|x\rangle \rightarrow |y\rangle$ for $\omega_q t = \pi/2$). We will see that only the component of $\vec{m}(t)$ in Eq. 10 that rotates also counter-clockwise will contribute significantly to the evolution of the driven system, while the clockwise component oscillates too fast and does not contribute.

Transforming the Hamiltonian into the rotating frame means that we apply the unitary transformation

$$U = e^{i\frac{\widetilde{\omega}t}{2}\sigma_z}$$

to the system state,

$$|\phi\rangle = U|\psi\rangle = e^{i\frac{\widetilde{\omega}t}{2}\sigma_z}|\psi\rangle \tag{12}$$

The Hamiltonian transforms accordingly,

$$UHU^\dagger \underbrace{U|\psi\rangle}_{|\phi\rangle} = i\hbar U\frac{d}{dt}\left(U^\dagger \underbrace{U|\psi\rangle}_{|\phi\rangle}\right) = i\hbar(U\dot{U}^\dagger + \frac{d}{dt}|\phi\rangle) \tag{13}$$

and therefore

$$\widetilde{H}|\phi\rangle = i\hbar\frac{d}{dt}|\phi\rangle \tag{14}$$

with

$$\widetilde{H} = UHU^\dagger - iU\dot{U}^\dagger. \tag{15}$$

To calculate $\widetilde{H}$ we compute the transformed Pauli matrices,

$$U\sigma_z U^\dagger = \sigma_z \tag{16}$$

$$U\sigma_x U^\dagger = \cos(\widetilde{\omega}t)\sigma_x - \sin(\widetilde{\omega}t)\sigma_y \tag{17}$$

$$U\sigma_y U^\dagger = \sin(\widetilde{\omega}t)\sigma_x + \cos(\widetilde{\omega}t)\sigma_y \tag{18}$$

as well as

$$U\dot{U}^\dagger = -i\frac{\widetilde{\omega}}{2}\sigma_z \tag{19}$$

These expression are then used to transform the laboratory frame Hamiltonian of the qubit,

$$H_{nonRWA} = \frac{\omega_q}{2}\sigma_z + \frac{\Omega}{2}\underbrace{[\cos(\omega t + \phi)\sigma_x + \sin(\omega t + \phi)\sigma_y]}_{A+} + \frac{\Omega}{2}\underbrace{[\cos(\omega t + \phi)\sigma_x - \sin(\omega t + \phi)\sigma_y]}_{A-}. \tag{20}$$

Explicitly, we obtain

$$A\pm = \cos(\omega t + \phi)\left(\cos(\widetilde{\omega}t)\sigma_x - \sin(\widetilde{\omega}t)\sigma_y\right) \pm \sin(\omega t + \phi)\left(\sin(\widetilde{\omega}t)\sigma_x + \cos(\widetilde{\omega}t)\sigma_y\right) \tag{21}$$

$$= \left[\cos(\omega t + \phi)\cos(\widetilde{\omega}t) \pm \sin(\omega t + \phi)\sin(\widetilde{\omega}t)\right]\sigma_x \tag{22}$$

$$+ \left[-\cos(\omega t + \phi)\sin(\widetilde{\omega}t) \pm \sin(\omega t + \phi)\cos(\widetilde{\omega}t)\right]\sigma_y \tag{23}$$

$$= \cos[(\omega \mp \widetilde{\omega}) + \phi]\sigma_x \pm \sin[(\omega \mp \widetilde{\omega})t + \phi]\sigma_y. \tag{24}$$

Choosing $\widetilde{\omega} = \omega$, i.e. a frame resonant with the drive we end up with

$$\begin{aligned} A+ &= \cos\phi\sigma_x + \sin\phi\sigma_y && (25)\\ A- &= \cos[2\omega t + \phi]\sigma_x - \sin[2\omega t + \phi]\sigma_y. && (26) \end{aligned}$$

The $A-$ terms are rotating with twice the drive frequency, whereas the $A+$ terms is static. The total Hamiltonian in the rotating frame (including the gauge term $U\dot{U}^\dagger$) reads

$$H_{rot} = \frac{\omega_q - \omega}{2}\sigma_z + \frac{\Omega}{2}\left(\cos\phi\sigma_x + \sin\phi\sigma_y\right) \equiv \frac{\delta}{2}\sigma_z + \frac{\Omega_x}{2}\sigma_x + \frac{\Omega_y}{2}\sigma_y. \tag{27}$$

This corresponds to a spin-1/2 particle in the effective magnetic field

$$\widetilde{m}(t) = \begin{pmatrix} \Omega_x \\ \Omega_y \\ \delta \end{pmatrix}.$$