Coplanar Waveguide Resonators for Circuit Quantum Electrodynamics

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We have designed and fabricated superconducting coplanar waveguide resonators with fundamental frequencies from 2 to 9 GHz and loaded quality factors ranging from a few hundreds to a several hundred thousands reached at temperatures of 20 mK. The loaded quality factors are controlled by appropriately designed input and output coupling capacitors. The measured transmission spectra are analyzed using both a lumped element model and a distributed element transmission matrix method. The experimentally determined resonance frequencies, quality factors and insertion losses are fully and consistently characterized by the two models for all measured devices. Such resonators find prominent applications in quantum optics and quantum information processing with superconducting electronic circuits and in single photon detectors and parametric amplifiers.

I. INTRODUCTION

Superconducting coplanar waveguide (CPW) resonators find a wide range of applications as radiation detectors in the optical, UV and X-ray frequency range, in parametric amplifiers, for magnetic field tunable resonators and in quantum optics and quantum information processing. In the recent past it has been experimentally demonstrated that a single microwave photon stored in a high quality CPW resonator can be coherently coupled to a superconducting quantum two-level system. This possibility has lead to a wide range of novel quantum optics experiments realized in an architecture now known as circuit quantum electrodynamics (QED). The circuit QED architecture is also successfully employed in quantum information and quantum optics experiments.

In this paper we discuss the use of CPWs in the context of quantum optics and quantum information processing. In the planar geometry of a capacitively coupled CPW resonator is sketched in Figure 1a. The resonator is formed of a center conductor of width $w = 10 \mu m$ separated from the lateral ground planes by a gap of width $s = 6.6 \mu m$. Resonators with various center conductor lengths $l$ between 8 and 29 mm aiming at fundamental frequencies $f_0$ between 2 and 9 GHz were designed. These structures are easily fabricated in optical lithography while providing sufficiently large vacuum fluctuations, a key ingredient for realizing strong coupling between photons and qubits in the circuit QED architecture. Moreover, CPW resonators with large internal quality factors of typically several hundred thousands can now be routinely realized.

In this paper we demonstrate that we are able to design, fabricate and characterize CPW resonators with well defined resonance frequency and coupled quality factors. The resonance frequency is controlled by the resonator length and its loaded quality factor is controlled by its capacitive coupling to input and output transmission lines. Strongly coupled (overcoupled) resonators with accordingly low quality factors are ideal for performing fast measurements of the state of a qubit integrated into the resonator. On the other hand, undercoupled resonators with large quality factors can be used to store photons in the cavity on a long time scale, with potential use as a quantum memory.

The paper is structured as follows. In Sec. II we discuss the chosen CPW device geometry, its fabrication and the measurement techniques used for characterization at microwave frequencies. The dependence of the CPW resonator frequency on the device geometry and its electrical parameters is analyzed in Sec. III. In Sec. IV the effect of the resonator coupling to an input/output line on its quality factor, insertion loss and resonance frequency is analyzed using a parallel LCR circuit model. This lumped element model provides simple approximations of the resonator properties around resonance and allows to develop an intuitive understanding of the device. We also make use of the transmission (or ABCD) matrix method to describe the full transmission spectrum of the resonators and compare its predictions to our experimental data. The characteristic properties of the higher harmonic modes of the CPW resonators are discussed in Sec. V.

II. DEVICE GEOMETRY, FABRICATION AND MEASUREMENT TECHNIQUE

The planar geometry of a capacitively coupled CPW resonator is sketched in Figure 1a. The resonator is formed of a center conductor of width $w = 10 \mu m$ separated from the lateral ground planes by a gap of width $s = 6.6 \mu m$. Resonators with various center conductor lengths $l$ between 8 and 29 mm aiming at fundamental frequencies $f_0$ between 2 and 9 GHz were designed. These structures are easily fabricated in optical lithography while providing sufficiently large vacuum fluctuations, a key ingredient for realizing strong coupling between photons and qubits in the circuit QED architecture. Moreover, CPW resonators with large internal quality factors of typically several hundred thousands can now be routinely realized.

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field strengths. The center conductor is coupled via gap- or finger capacitors to the input and output transmission lines. For small coupling capacitances gap capacitors of widths $w_g = 10$ to $50 \mu m$ have been realized. To achieve larger coupling, finger capacitors formed by from one up to eight pairs of fingers of length $l_f = 100 \mu m$, width $w_f = 3.3 \mu m$ and separation $s_f = 3.3 \mu m$ have been designed and fabricated, see Fig. 1.

The resonators are fabricated on high resistivity, undoped, (100)-oriented, thermally oxidized two inch silicon wafers. The oxide thickness is $h_2 = 550 \text{ nm} \pm 50 \text{ nm}$ determined by SEM inspection. The bulk resistivity of the Si wafer is $\rho > 3000 \text{ \Omega cm}$ determined at room temperature in a van-der-Pauw measurement. The total thickness of the substrate is $h_1 = 500 \mu m \pm 25 \mu m$. A cross-sectional sketch of the CPW resonator is shown in Fig. 1b.

The resonators were patterned in optical lithography using a one micron thick layer of the negative tone resist ma-N 1410. The substrate was subsequently metallized with a $t = 200 \text{ nm} \pm 5 \text{ nm}$ thick layer of Al, electron beam evaporated at a rate of $5 \text{ A/sec}$ and lifted-off in $50^\circ C$ hot acetone. Finally, all structures were diced into $2 \text{ mm} \times 7 \text{ mm}$ chips, each containing an individual resonator. The feature sizes of the fabricated devices deviate less than $100 \text{ nm}$ from the designed dimensions as determined by SEM inspection indicating a good control over the fabrication process.

Altogether, more than 80 Al CPW resonators covering a wide range of different coupling strengths were designed and fabricated. More than 30 of these devices were carefully characterized at microwave frequencies. Figure 2 shows optical microscope images of the final Al resonators with different finger and gap capacitors.

Using a 40 GHz vector network analyzer, $S_{21}$ transmission measurements of all resonators were performed in a pulse-tube based dilution refrigerator system at temperatures of $20 \text{ mK}$. The measured transmission spectra are plotted in logarithmic units (dB) as $20 \log_{10}|S_{21}|$. High $Q$ resonators were measured using a $32 \text{ dB}$ gain high electron mobility transistor (HEMT) amplifier with noise temperature of $\sim 5 \text{ K}$ installed at the $4 \text{ K}$ stage of the refrigerator as well as one or two room temperature amplifiers with $35 \text{ dB}$ gain each. Low $Q$ resonators were characterized without additional amplifiers.

The measured $Q$ of undercoupled devices can vary strongly with the power applied to the resonator. In our measurements of high $Q$ devices the resonator transmission spectrum looses its Lorentzian shape at drive powers above approximately $-70 \text{ dBm}$ at the input port of the resonator due to non-linear effects. At low drive powers, when dielectric resonator losses significantly depend on the photon number inside the cavity, measured quality factors may be substantially reduced. We acquired $S_{21}$ transmission spectra at power levels chosen to result in the highest measurable quality factors, i.e. at high enough powers to minimize dielectric loss but low enough to avoid non-linearities. This approach has been chosen to be able to focus on geometric properties of the resonators.

### III. BASIC RESONATOR PROPERTIES

A typical transmission spectrum of a weakly gap capacitor coupled ($w_g = 10 \mu m$) CPW resonator of length $l = 14.22 \text{ mm}$ is shown in Figure 3a. The spectrum clearly displays a Lorentzian lineshape of width $\delta f$ centered at the resonance frequency $f_0$. Figure 3b shows measured resonance frequencies $f_0$ for resonators of different length $l$, all coupled via gap capacitors of widths $w_g = 10 \mu m$. Table I lists the respective values for $l$ and $f_0$. For these small capacitors the frequency shift induced by coupling can be neglected, as discussed in a later section. In this case the resonator’s fundamental frequency $f_0$ is given by

$$f_0 = \frac{c}{\sqrt{\epsilon_{\text{eff}}}} \frac{1}{2l},$$

(1)
FIG. 3: (Color online) (a) Transmission spectrum of a 4.7 GHz resonator. Data points (blue) were fitted (black) with a Lorentzian line. (b) Measured $f_0$ (red points) of several resonators coupled via $w_g = 10 \mu m$ gap capacitors with different $l$ together with a fit (blue line) to the data using Eq. (1) as fit function and $\epsilon_{eff}$ as fit parameter.

TABLE I: Designed values for resonator lengths $l$ and measured resonance frequencies $f_0$, corresponding to the data shown in Fig. 3.

<table>
<thead>
<tr>
<th>$f_0$ (GHz)</th>
<th>$l$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3430</td>
<td>28.449</td>
</tr>
<tr>
<td>3.5199</td>
<td>18.970</td>
</tr>
<tr>
<td>4.6846</td>
<td>14.220</td>
</tr>
<tr>
<td>5.8491</td>
<td>11.380</td>
</tr>
<tr>
<td>7.0162</td>
<td>9.4800</td>
</tr>
<tr>
<td>8.1778</td>
<td>8.1300</td>
</tr>
</tbody>
</table>

TABLE II: Properties of the different CPW resonators whose transmission spectra are shown in Fig. 4. $C_n$ denotes the simulated coupling capacitances, $f_0$ is the measured resonance frequency and $Q_L$ is the measured quality factor.

<table>
<thead>
<tr>
<th>ID</th>
<th>Coupling</th>
<th>$C_n$ (fF)</th>
<th>$f_0$ (GHz)</th>
<th>$Q_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8 + 8 finger</td>
<td>56.4</td>
<td>2.2678</td>
<td>3.7 \cdot 10^2</td>
</tr>
<tr>
<td>B</td>
<td>7 + 7 finger</td>
<td>48.6</td>
<td>2.2763</td>
<td>4.9 \cdot 10^2</td>
</tr>
<tr>
<td>C</td>
<td>6 + 6 finger</td>
<td>42.9</td>
<td>2.2848</td>
<td>7.5 \cdot 10^2</td>
</tr>
<tr>
<td>D</td>
<td>5 + 5 finger</td>
<td>35.4</td>
<td>2.2943</td>
<td>1.1 \cdot 10^3</td>
</tr>
<tr>
<td>E</td>
<td>4 + 4 finger</td>
<td>26.4</td>
<td>2.3086</td>
<td>1.7 \cdot 10^3</td>
</tr>
<tr>
<td>F</td>
<td>3 + 3 finger</td>
<td>18.0</td>
<td>2.3164</td>
<td>3.9 \cdot 10^3</td>
</tr>
<tr>
<td>G</td>
<td>2 + 2 finger</td>
<td>11.3</td>
<td>2.3259</td>
<td>9.8 \cdot 10^3</td>
</tr>
<tr>
<td>H</td>
<td>1 + 1 finger</td>
<td>3.98</td>
<td>2.3343</td>
<td>7.5 \cdot 10^4</td>
</tr>
<tr>
<td>I</td>
<td>10 \mu m gap</td>
<td>0.44</td>
<td>2.3430</td>
<td>2.0 \cdot 10^5</td>
</tr>
<tr>
<td>J</td>
<td>20 \mu m gap</td>
<td>0.38</td>
<td>2.3448</td>
<td>2.0 \cdot 10^5</td>
</tr>
<tr>
<td>K</td>
<td>30 \mu m gap</td>
<td>0.32</td>
<td>2.3459</td>
<td>2.3 \cdot 10^5</td>
</tr>
<tr>
<td>L</td>
<td>50 \mu m gap</td>
<td>0.24</td>
<td>2.3464</td>
<td>2.3 \cdot 10^5</td>
</tr>
</tbody>
</table>

Here, $c/\sqrt{\epsilon_{eff}} = v_{ph}$ is the phase velocity depending on the velocity of light in vacuum $c$ and the effective permittivity $\epsilon_{eff}$ of the CPW line. $\epsilon_{eff}$ is a function of the waveguide geometry and the relative permittivities $\epsilon_1$ and $\epsilon_2$ of substrate and the oxide layer, see Fig. 1b. Furthermore, $2l = \lambda_0$ is the wavelength of the fundamental resonator mode. The length dependence of the measured resonance frequencies $f_0$ of our samples is well described by Eq. (1) with the effective dielectric constant $\epsilon_{eff} = 5.05$, see Fig. 3b.

The phase velocity $v_{ph} = 1/\sqrt{L_C}\omega$ of electromagnetic waves propagating along a transmission line depends on the capacitance $C_L$ and inductance $L_L$ per unit length of the line. Using conformal mapping techniques the geometric contribution to $L_L$ and $C_L$ of a CPW line is found to be

$$L_L = \frac{\mu_0 K(k_0)}{4} \frac{K(k_0)}{K(k_0)}.$$  \hspace{1cm} (2)

$$C_L = 4\epsilon_0\epsilon_{eff} \frac{K(k_0)}{K(k_0)}.$$  \hspace{1cm} (3)

Here, $K$ denotes the complete elliptic integral of the first kind with the arguments

$$k_0 = \frac{w}{w + 2s},$$  \hspace{1cm} (4)

$$k_0' = \sqrt{1 - k_0^2}.$$  \hspace{1cm} (5)

For non magnetic substrates ($\mu_{eff} = 1$) and neglecting kinetic inductance for the moment $L_L$ is determined by the CPW geometry only. $C_L$ depends on the geometry and $\epsilon_{eff}$. Although analytical expressions for $\epsilon_{eff}$ exist for double layer substrates deduced from conformal mapping, the accuracy of these calculations depends sensitively on the ratio between substrate layer thicknesses and the dimensions of the CPW cross-section and does not lead to accurate predictions for our parameters. Therefore, we have calculated $C_L \approx 1.27 \cdot 10^{-10}$ Fm$^{-1}$ using a finite element electromagnetic simulation and values $\epsilon_1 = 11.6$ (see Ref. 37) for silicon and $\epsilon_2 = 3.78$ (see Ref. 37) for silicon oxide for our CPW geometry and substrate. From this calculation we find $\epsilon_{eff} \approx 5.22$ which deviates only by about 3% from the value extracted from our measurements. The characteristic impedance of a CPW is then given by $Z_0 = \sqrt{L_C/C_L}$ which results in a value of 59.7\,\Omega for our geometry. This value deviates from the usually chosen value of 50\,\Omega as the original design was optimized for a different substrate material.

In general, for superconductors the inductance $L_L$ is the sum of the temperature independent geometric (magnetic) inductance $L_L^m$ and the temperature dependent kinetic inductance $L_L^k$ (see Ref. 38). For superconductors, $L_L^k$ refers to the inertia of moving Cooper pairs and can contribute significantly to $L_L$ since resistivity is suppressed and thus charge carrier relaxation times are large. According to Ref. 35, $L_L^k$ scales with $\lambda^2(T)$, where $\lambda(T)$ is the temperature dependent London penetration depth which can be approximated as $\lambda(T) = 1.05 \cdot 10^{-3} \sqrt{\rho(T)/\rho(0)} \sqrt{K m/\Omega}$ at zero temperature in the
local and dirty limits. In the dirty (local) limit the mean free path of electrons $\ell_{\text{mf}}$ is much less than the coherence length $\xi_0 = \hbar v_f / \pi \Delta(0)$, where $v_f$ is the Fermi velocity of the electrons and $\Delta(0)$ is the superconducting gap energy at zero temperature. The clean (nonlocal) limit occurs when $\ell_{\text{mf}}$ is much larger than $\xi_0$ (see Ref. 39). $T_c = 1.23 \, \text{K}$ is the critical temperature of our thin film aluminum and $\rho(T_c) = 2.06 \cdot 10^{-8} \, \Omega \, \text{m}$ is the normal state resistivity at $T = T_c$. $T_c$ and $\rho(T)$ were determined in a four-point measurement of the resistance of a lithographically patterned Al thin film meander structure from the same substrate in dependence on temperature. The resulting residual resistance ratio ($\text{RRR}_{300K/1.3K}$) is 8.6. Since our measurements were performed at temperatures well below $T_c$, $\lambda = \lambda(0)$ approximately holds and we find $\lambda(0) \approx 43 \, \text{nm}$ for our Al thin films (compared to a value of 40 nm, given in Ref. 40). Using the above approximation shows that $L_r^m$ is about two orders of magnitude smaller than $L_r^m = 4.53 \cdot 10^{-7} \, \text{H} \, \text{m}^{-1}$ legitimating the assumption $L_r \approx L_r^0$ made in Eq. (2). Kinetic inductance effects in Niobium resonators are also analyzed in Ref. 25.

IV. INPUT/OUTPUT COUPLING

To study the effect of the capacitive coupling strength on the microwave properties of CPW resonators, twelve 2.3 GHz devices, symmetrically coupled to input/output lines with different gap and finger capacitors have been characterized, see Table II for a list of devices. The measured transmission spectra are shown in Fig. 4. The left hand part of Fig. 4 depicts spectra of resonators coupled via finger capacitors having 8 down to one pairs of fingers (devices A to H). The right hand part of Fig. 4 shows those resonators coupled via gap capacitors with gap widths of $w_g = 10, 20, 30$ and 50 $\mu\text{m}$ (devices I to L) respectively. The coupling capacitance continuously decreases from device A to device L. The nominal values for the coupling capacitance $C_s$ obtained from EM-simulations for the investigated substrate properties and geometry are listed in Table II. The resonance frequency $f_0$ and the measured quality factor $Q_L = f_0 / \delta f$ of the respective device is obtained by fitting a Lorentzian line shape

$$F_{\text{Lor}}(f) = A_0 \frac{\delta f}{(f - f_0)^2 + \delta f^2 / 4},$$

(6)

to the data, see Fig. 3a, where $\delta f$ is the full width half maximum of the resonance. With increasing coupling capacitance $C_s$, Fig. 4 shows a decrease in the measured (loaded) quality factor $Q_L$ and an increase in the peak transmission, as well as a shift of $f_0$ to lower frequencies. In the following, we demonstrate how these characteristic resonator properties can be fully understood and modeled consistently for the full set of data.

A transmission line resonator is a distributed device with voltages and currents varying in magnitude and phase over its length. The distributed element representation of a symmetrically coupled resonator is shown in Fig. 5a. $R_L$, $L_L$ and $C_L$ denote the resistance, inductance and capacitance per unit length, respectively. According to Ref. 41 the impedance of a TL resonator is given by

$$Z_{\text{TL}} = Z_0 \frac{1 + i \tan \beta \tanh \alpha l}{\tan \alpha l + i \tan \beta l} \left(7\right)$$

(7)

$$\approx Z_0 \frac{Z_0}{\alpha l + i \frac{\omega}{\omega_n}(\omega - \omega_n)}.$$  

(8)

$\alpha$ is the attenuation constant and $\beta = \omega_n / v_{ph}$ is the phase constant of the TL. The approximation in Eq. (8) holds when assuming small losses ($\alpha l \ll 1$) and for $\omega$ close to $\omega_n$. Here, $\omega_n = n \omega_0 = 1 / \sqrt{L_0 C}$ is the angular frequency of the $n$-th mode, where $n$ denotes the resonance mode number ($n = 1$ for the fundamental mode).

Around resonance, the properties of a TL resonator can be approximated by those of a lumped element, parallel
LCR oscillator, as shown in Fig. 5b, with impedance
\[ Z_{\text{LCR}} = \left( \frac{1}{i\omega L_n} + i\omega C + \frac{1}{R} \right)^{-1} \]
\[ \approx \frac{R}{1 + 2iRC(\omega - \omega_n)}, \]  
and characteristic parameters
\[ L_n = \frac{2Ll}{n^2\pi^2}, \]
\[ C = \frac{Cld}{2i}, \]
\[ R = \frac{Z_0}{\alpha l}. \]
The approximation Eq. (10) is valid for \( \omega \approx \omega_n \). The LCR model is useful to get an intuitive understanding of the resonator properties. It simplifies analyzing the effect of coupling the resonator to an input/output line on the quality factor and on the resonance frequency as discussed in the following.

The (internal) quality factor of the parallel LCR oscillator is defined as
\[ Q_{\text{int}} = \frac{1}{\omega_n RC \kappa}. \]
The quality factor \( Q_L \) of the resonator coupled with capacitance \( C_{\kappa} \) to the input and output lines with impedance \( Z_0 \) is reduced due to the resistive loading. Additionally, the frequency is shifted because of the capacitive loading of the resonator due to the input/output lines. To understand this effect the series connection of \( C_{\kappa} \) and \( R_l \) can be transformed into a Norton equivalent parallel connection of a resistor \( R^* \) and a capacitor \( C^* \), see Figs. 5b, c, with
\[ R^* = \frac{1 + \omega_n^2 C_{\kappa}^2 R_l^2}{\omega_n^2 C_{\kappa}^2 R_l}, \]  
\[ C^* = \frac{C_{\kappa}}{1 + \omega_n^2 C_{\kappa}^2 R_l^2}. \]
The small capacitor \( C_{\kappa} \) transforms the \( R_l = 50 \Omega \) load into the large impedance \( R^* = R_l/k^2 \) with \( k = \omega_n C_{\kappa} R_l \ll 1 \). For symmetric input/output coupling the loaded quality factor for the parallel combination of \( R \) and \( R^*/2 \) is
\[ Q_L = \frac{\omega_n^* C + 2C^*}{1/R + 2/R^*} \approx \frac{\omega_n^* C}{1/R + 2/R^*} \]
with the \( n \)-th resonance frequency shifted by the capacitive loading due to the parallel combination of \( C \) and \( 2C^* \)
\[ \omega_n^* = \frac{1}{\sqrt{L_n(C + 2C^*)}}. \]  
For \( \omega_n^* \approx \omega_n \) with \( C + 2C^* \approx C \), the Norton equivalent expression for the loaded quality factor \( Q_L \) is a parallel combination of the internal and external quality factors
\[ \frac{1}{Q_L} = \frac{1}{Q_{\text{int}}} + \frac{1}{Q_{\text{ext}}}, \]
with
\[ Q_{\text{int}} = \omega_n^* RC = \frac{n\pi}{2\alpha l}, \]
\[ Q_{\text{ext}} = \frac{\omega_n^* R^* C^*}{2}. \]
The measured loaded quality factor \( Q_L \) for devices A to L is plotted vs. the coupling capacitance in Fig. 6a. \( Q_L \) is observed to be constant for small coupling capacitances and decreases for large ones. In the overcoupled regime \( (Q_{\text{ext}} \ll Q_{\text{int}}) \), \( Q_L \) is governed by \( Q_{\text{ext}} \) which is well approximated by \( C/2\omega_n R_l C_{\kappa}^2 \), see dashed line in Fig. 6. Thus, in the overcoupled regime the loaded quality factor \( Q_L \propto C_{\kappa}^{-2} \) can be controlled by the choice of the coupling capacitance. In the undercoupled limit \( (Q_{\text{ext}} \gg Q_{\text{int}}) \) however, \( Q_L \) saturates at the internal quality factor \( Q_{\text{int}} = 2.3 \cdot 10^5 \) determined by the intrinsic losses of the resonator, see horizontal dashed line in Fig. 6a.

Radiation losses are expected to be small in CPW resonators\(^{43} \), resistive losses are negligible well below the critical temperature \( T_c \) of the superconductor\(^{26} \) and at frequencies well below the superconducting gap. We believe that dielectric losses limit the internal quality factor of our devices, as discussed in References 33 and 28.

Using Eqs. (14, 19, 21), \( C_{\kappa} \) has been extracted from the measured value of \( Q_{\text{int}} \approx 2.3 \cdot 10^5 \) and the measured loaded quality factors \( Q_L \) of the overcoupled devices A to H, see Fig. 7. The experimental values of \( C_{\kappa} \) are in good agreement with the ones found from finite element calculations, listed in table II, with a standard deviation of about 4%.

The insertion loss
\[ L_0 = -20 \log \left( \frac{g}{g + 1} \right) \text{ dB} \]  
(22)
of a resonator, i.e. the deviation of peak transmission from unity, is dependent on the ratio of the internal to the external quality factor which is also called the coupling coefficient $g = Q_{\text{int}}/Q_{\text{ext}}$ (see Ref. 41). The measured values of $L_0$ as extracted from Fig. 4 are shown in Fig. 6b. For $g > 1$ (large $C_\kappa$) the resonator is overcoupled and shows near unit transmission ($L_0 = 0$). The resonator is said to be critically coupled for $g = 1$. For $g < 1$ (small $C_\kappa$) the resonator is undercoupled and the transmission is significantly reduced. In this case $L_0$ is well approximated by $-20 \log(2\omega_0 Q_{\text{int}} R_L C_\kappa^2/C)$, see dashed line in Fig. 6b, as calculated from Eqs. (14, 21, 22). $Q_{\text{ext}}$ and $Q_{\text{int}}$ can be determined from $Q_L$ and $L_0$ using Eqs. (19, 22), thus allowing to roughly estimate internal losses even of an overcoupled cavity.

For the overcoupled devices A to H the coupling induced resonator frequency shift as extracted from Fig. 4 is in good agreement with calculations based on Eqs. (15, 18), see Fig. 7b. For $C^* \approx C_\kappa$ and $C \gg C_{\kappa}$ one can Taylor-approximate $\omega_0^* = \omega_0(1 - C^*/C)$, As a result the relative resonator frequency shift is $(\omega_0^* - \omega_0)/\omega_0 = -C/\kappa$ for symmetric coupling. Figure 7b shows the expected linear dependence with a maximum frequency shift of about 3% over a range of 60 fF in $C_{\kappa}$.

As an alternative method to the LCR model which is only an accurate description near resonance we have analyzed our data using the transmission matrix method\(^4\). Using this method the full transmission spectrum of the CPW resonator can be calculated. However, because of the mathematical structure of the model it is more involved to gain intuitive understanding of the CPW devices.

All measured $S_{21}$ transmission spectra are consistently fit with a single set of parameters, see Fig. 4. The transmission or ABCD matrix of a symmetrically coupled TL is defined by the product of an input-, a transmission-, and an output matrix as

$$
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
= \begin{bmatrix} 1 & Z_{\text{in}} \\ 0 & 1 \end{bmatrix}
\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}
\begin{bmatrix} 1 & Z_{\text{out}} \\ 0 & 1 \end{bmatrix}.
$$

(23)

with input/output impedances $Z_{\text{in/out}} = 1/\omega C_{\kappa}$ and the transmission matrix parameters

$$
t_{11} = \cosh(\gamma l),
$$

(24)

$$
t_{12} = Z_0 \sinh(\gamma l),
$$

(25)

$$
t_{21} = 1/Z_0 \sinh(\gamma l),
$$

(26)

$$
t_{22} = \cosh(\gamma l).
$$

(27)

Here, $\gamma = \alpha + i \beta$ is the TL wave propagation coefficient. The resonator transmission spectrum is then defined by the ABCD matrix components as

$$
S_{21} = \frac{2}{A + B/R_L + C R_L + D}.
$$

(28)

Here, $R_L$ is the real part of the load impedance, accounting for outer circuit components. $\alpha$ is determined by $Q_{\text{int}}$ and $l$ and $\beta$ depends on $\epsilon_{\text{eff}}$ as discussed before. According to Eqs. (2, 3) $Z_0$ is determined by $\epsilon_{\text{eff}}$, $w$ and $s$. The attenuation constant is $\alpha \sim 2.4 \cdot 10^{-4}$ m$^{-1}$ as determined from $Q_{\text{int}} \sim 2.3 \cdot 10^5$.\(^4\)
For gap capacitor coupled devices, the measured data fits very well, see Fig. 4, to the transmission spectrum calculated using the ABCD matrix method with $\tau_{\text{eff}} = 5.05$, already obtained from the measured dependence of $f_0$ on the resonator length, see Fig. 3. For finger capacitor coupled structures however, see Fig. 1a, approximately 40% of the length of each 100 $\mu$m finger has to be added to the length $l$ of the bare resonators in order to obtain good fits to the resonance frequency $f_0$. This result is independent of the number of fingers. The ABCD matrix model describes the full transmission spectra of all measured devices very well with a single set of parameters, see Fig. 4.

V. HARMONIC MODES

So far we have only discussed the properties of the fundamental resonance frequency of any of the measured resonators. A full transmission spectrum of the overcoupled resonator D, including 5 harmonic modes, is shown in Fig. 8. The measured spectrum fits well to the ABCD matrix model for the fundamental frequency and also for higher cavity modes, displaying a decrease of the loaded quality factor with harmonic number. The dependence of the measured quality factor $Q_L$ on the mode number $n$ is in good agreement with Eqs. (19, 21) and scales approximately as $C/2n\omega_0 R_0 C_L^2$.

VI. CONCLUSIONS

In summary, we have designed and fabricated symmetrically coupled coplanar waveguide resonators over a wide range of resonance frequencies and coupling strengths. We demonstrate that loaded quality factors and resonance frequencies can be controlled and that the LCR- and ABCD matrix models are in good agreement with measured data for fundamental and harmonic modes. In the case of resonators coupled via finger capacitors simulated values for $C_s$ deviate by only about 4%. About 40% of the capacitor finger length has to be added to the total resonator length to obtain a good fit to the resonance frequency.

The resonator properties discussed above are consistent with those obtained from measurements of additional devices with fundamental frequencies of 3.5, 4.7, 5.8, 7.0 and 8.2 GHz. The experimental results presented in this paper were obtained for Al based resonators on an oxidized silicon substrate. The methods of analysis should also be applicable to CPW devices fabricated on different substrates and with different superconducting materials. The good understanding of geometric and electrical properties of CPW resonators will certainly foster further research on their use as radiation detectors, in quantum electrodynamics and quantum information processing applications.

FIG. 8: Measured quality factors for the overcoupled resonator D vs. mode number $n$ (red points) together with the prediction of the mapped LCR model given by Eqs. (18, 21) (solid blue line). The inset shows the $S_{21}$ transmission spectrum of resonator D with fundamental mode and harmonics. The measured data (blue) is compared to the $S_{21}$ spectrum (black) obtained by the ABCD matrix method.

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