2.7.6 Super Dense Coding

task: Try to transmit two bits of classical information between Alice (A) and Bob (B) using only one qubit.

- As Alice and Bob are living in a quantum world they are allowed to use one pair of entangled qubits that they have prepared ahead of time.

protocol:
A) Alice and Bob each have one qubit of an entangled pair in their possession

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right) \]

B) Alice does a quantum operation on her qubit depending on which 2 classical bits she wants to communicate
C) Alice sends her qubit to Bob
D) Bob does one measurement on the entangled pair

shared entanglement

local operations

send Alices qubit to Bob

Bob measures
<table>
<thead>
<tr>
<th>Bits to be transferred:</th>
<th>Alice's operation</th>
<th>Resulting 2-qubit state</th>
<th>Bob's measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>$I_1$</td>
<td>$I_1 \left</td>
<td>\psi \right\rangle = \frac{1}{\sqrt{2}} \left(</td>
</tr>
<tr>
<td>01</td>
<td>$Z_1$</td>
<td>$Z_1 \left</td>
<td>\psi \right\rangle = \frac{1}{\sqrt{2}} \left(</td>
</tr>
<tr>
<td>10</td>
<td>$X_1$</td>
<td>$X_1 \left</td>
<td>\psi \right\rangle = \frac{1}{\sqrt{2}} \left(</td>
</tr>
<tr>
<td>11</td>
<td>$\gamma_1$</td>
<td>$\gamma_1 \left</td>
<td>\psi \right\rangle = \frac{1}{\sqrt{2}} \left(</td>
</tr>
</tbody>
</table>

**Comments:**
- Two qubits are involved in protocol but Alice only interacts with one and sends only one along her quantum communications channel.
- Two bits cannot be communicated sending a single classical bit along a classical communications channel.

2.7.7 Experimental demonstration of super dense coding using photons
Generating polarization entangled photon pairs using **Parametric Down Conversion**:

- 1 UV-photon $\rightarrow$ 2 "red" photons
- Conservation of energy
  \[ \omega_p = \omega_s + \omega_i \]
- Conservation of momentum
  \[ \vec{k}_p = \vec{k}_s + \vec{k}_i \]
- Polarisationskorrelationen (typ ll)

Optically nonlinear medium: BBO (BaB$_2$O$_4$) beta barium borate

\[ |\Psi^-(\text{V}H)\rangle = \frac{1}{\sqrt{2}} (|H\rangle_V - |V\rangle_H) \]
**Half Wave Plate**

*Figure 8.46* A half-wave plate.

between $n_e$ and $n_o$, is a bit too large for convenience. On the other hand, quartz with its much smaller birefringence is frequently used, but it has no natural cleavage planes and must be cut, ground, and polished.
Bell state measurement

\[ \psi^- = \frac{1}{\sqrt{2}} \left( |HV\rangle - |VH\rangle \right) \]  
\[ \psi^+ = \frac{1}{\sqrt{2}} \left( |HV\rangle + |VH\rangle \right) \]
\[ \phi^+ = \frac{1}{\sqrt{2}} \left( |HH\rangle + |VV\rangle \right) \]
\[ \phi^- = \frac{1}{\sqrt{2}} \left( |HH\rangle - |VV\rangle \right) \]

\( H = \) horizontal polarization  
\( V = \) vertical polarization

2.9 Quantum Teleportation

Task: Alice wants to transfer an unknown quantum state $\psi$ to Bob only using one entangled pair of qubits and classical information as a resource.

note:
- Alice does not know the state to be transmitted
- Even if she knew it the classical amount of information that she would need to send would be infinite.

The teleportation circuit:

![Teleportation Circuit Diagram]

2.9.1 How does it work?

(1) \[ |\psi\rangle \otimes \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \left( \alpha |000\rangle + \alpha |011\rangle + \beta |100\rangle + \beta |111\rangle \right) \]

CNOT between qubit to be teleported and one bit of the entangled pair:

(2) \[ \xrightarrow{\text{CNOT}_{12}} \frac{1}{\sqrt{2}} \left( \alpha |000\rangle + \alpha |101\rangle + \beta |110\rangle + \beta |111\rangle \right) \]

Hadamard on qubit to be teleported:

(3) \[ \xrightarrow{H_1} \frac{1}{2} \left[ \left( \alpha |00\rangle (\alpha |0\rangle + \beta |1\rangle) + |10\rangle (\alpha |0\rangle - \beta |1\rangle) \right) \right. \]
\[ + \left. |01\rangle (\alpha |1\rangle + \beta |0\rangle) + |11\rangle (\alpha |1\rangle - \beta |0\rangle) \right] \]

measurement of qubit 1 and 2, classical information transfer and single bit manipulation on target qubit 3:

(4) \[ \xrightarrow{M_1 \otimes M_2} \begin{align*} 
\rho_{00} &= \frac{1}{4} ; & |\psi_3\rangle &= \alpha |0\rangle + \beta |1\rangle \\
\rho_{10} &= \frac{1}{4} ; & |\psi_3\rangle &= \alpha |0\rangle - \beta |1\rangle \\
\rho_{01} &= \frac{1}{4} ; & |\psi_3\rangle &= \alpha |1\rangle + \beta |0\rangle \\
\rho_{11} &= \frac{1}{4} ; & |\psi_3\rangle &= \alpha |1\rangle - \beta |0\rangle 
\end{align*} \]

(5) \[ \xrightarrow{I} \]

(6) \[ \xrightarrow{2} \]

(7) \[ \xrightarrow{X} \]

(8) \[ \xrightarrow{X, 2} \]
2.9.2 (One) Experimental Realization of Teleportation using Photon Polarization:

- parametric down conversion (PDC)
- source of entangled photons
- qubits are polarization encoded

Experimental Implementation

start with states

\[ |\psi_1\rangle = \alpha |H\rangle + \beta |V\rangle \]

\[ |\psi_{2,3}\rangle = \frac{1}{\sqrt{2}} \left( |H\rangle |V\rangle - |V\rangle |H\rangle \right) \]

combine photon to be teleported (1) and one photon of entangled pair (2) on a 50/50 beam splitter (BS) and measure (at Alice) resulting state in Bell basis.

analyze resulting teleported state of photon (3) using polarizing beam splitters (PBS) single photon detectors

- polarizing beam splitters (PBS) as detectors of teleported states
teleportation papers for you to present:

**Experimental Realization of Teleporting an Unknown Pure Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels**
D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu

**Unconditional Quantum Teleportation**
A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik
Abstract » Full Text » PDF »

**Complete quantum teleportation using nuclear magnetic resonance**
M. A. Nielsen, E. Knill, R. Laflamme
Nature 396, 52 - 55 (05 Nov 1998) Letters to Editor
Abstract | Full Text | PDF | Rights and permissions | Save this link

**Deterministic quantum teleportation of atomic qubits**
Nature 429, 737 - 739 (17 Jun 2004) Letters to Editor
Abstract | Full Text | PDF | Rights and permissions | Save this link

**Deterministic quantum teleportation with atoms**
Nature 429, 734 - 737 (17 Jun 2004) Letters to Editor
Abstract | Full Text | PDF | Rights and permissions | Save this link

**Quantum teleportation between light and matter**
Jacob F. Sherson, Hanna Krauter, Rasmus K. Olsson, Brian Julsgaard, Klemens Hammerer, Ignacio Cirac, Eugene S. Polzik
Nature 443, 557 - 560 (05 Oct 2006) Letters to Editor
Full Text | PDF | Rights and permissions | Save this link
John Bell's thought experiment

- Charlie simultaneously prepares two particles having physical properties \( Q, R, S, T \) and gives one particle each to Alice and Bob.

- Alice measures the properties \( Q \) and \( R \) of her particle with the possible outcomes \( q = \pm 1 \) and \( r = \pm 1 \).

- Bob simultaneously measures the properties \( S \) and \( T \) of his particles with the possible outcomes \( s = \pm 1 \) and \( t = \pm 1 \).

consider the quantity:

\[
QS + RS + RT - QT
\]

\[
= (R+Q)S + (R-Q)T = \pm 2
\]

\[
\text{since } R+Q = 0 \text{ or } R-Q = 0
\]

the probability of the system being in state \( Q = q, R = r, S = s, T = t \)

is given by:

\[
\rho(q,r,s,t)
\]

we also denote \( E(x) \) as the mean of the quantity \( x \)

\[
E(x) = \sum_{q,r,s,t} \rho(q,r,s,t) x
\]

Now, Alice and Bob perform measurements on the two particles and record their outcomes. Then they meet up and perform the multiplications (e.g. \( q s \)) and calculate the average values \( E(QS) \).
What are the possible outcomes of measuring the quantity $E(QS+RS+RT-QT)$?

find an upper bound:

$$E(QS + RS + RT - QT) = \sum_{q, r, s, t} \rho(q, r, s, t) \left( QS + RS + RT - QT \right) \leq 2$$

$$\leq \sum_{q, r, s, t} \rho(q, r, s, t) = 1$$

also:

$$E(QS + RS + RT - QT) = \sum_{q, r, s, t} \rho(q, r, s, t) \left( QS + RS + RT - QT \right)$$

$$= E(QS) + E(RS) + E(RT) - E(QT) \leq 2$$

Bell inequality
measure this quantity for a Bell state:

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |10\rangle - |11\rangle \right) \]

Alice measures:
\[ Q = Z_1 \]
\[ R = X_1 \]

Bob measures:
\[ S = \frac{1}{\sqrt{2}} \left( -Z_2 - X_2 \right) \]
\[ T = \frac{1}{\sqrt{2}} \left( Z_2 - X_2 \right) \]

determine expectation values of joint measurements:

\[
\langle QS \rangle = \langle \psi | (Z_1 \frac{1}{\sqrt{2}} (-Z_2 - X_2)) | \psi \rangle = \frac{1}{\sqrt{2}} \langle \psi | -Z_1 Z_2 - Z_1 X_2 | \psi \rangle
\]
\[
= \frac{1}{\sqrt{2}} \frac{1}{2} \left( \langle 01 | - \langle 10 | \right) \left[ \left( |10\rangle - |11\rangle \right) - \left( |10\rangle + |11\rangle \right) \right] = \frac{1}{\sqrt{2}} \frac{1}{2} (1+1)
\]
\[
= \frac{1}{\sqrt{2}}
\]

\[
\langle RS \rangle = \frac{1}{\sqrt{2}} ; \quad \langle RT \rangle = \frac{1}{\sqrt{2}} ; \quad \langle QT \rangle = -\frac{1}{\sqrt{2}}
\]

determine value of Bell inequality:

\[
\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2 \sqrt{2} > 2
\]

Bell states maximally violate the Bell inequality!
Experimental violation of Bell Inequality (Alain Aspect):

generation of polarization entangled photons:

\[ J = 0 \]

\[ \Psi(v_1, v_2) = \frac{1}{\sqrt{2}} \left( |x, x\rangle + |y, y\rangle \right) \]

setup:

\[ N_{++}(a, b), N_{+-}(a, b), N_{-+}(a, b), N_{--}(a, b) \]

measure coincidences and calculate correlation coefficient:

\[ E(a, b) = \frac{N_{++}(a, b) - N_{+-}(a, b) - N_{-+}(a, b) + N_{--}(a, b)}{N_{++}(a, b) + N_{+-}(a, b) + N_{-+}(a, b) + N_{--}(a, b)} \]

if \((a, b) = 0\) (parallel polarizers) then \(E(a, b) = 1\), i.e. perfect correlation of results
quantum mechanical prediction:

\[
|\Psi(v_1, v_2)\rangle = \frac{1}{\sqrt{2}} \{ |x, x\rangle + |y, y\rangle \}
\]

probability of individual photon measurements

\[
P_+(a) = P_-(a) = \frac{1}{2} \quad ; \quad P_+(b) = P_-(b) = \frac{1}{2}
\]

probabilities of joint measurements on both photons:

\[
P_{++}(a, b) = P_{--}(a, b) = \frac{1}{2} \cos^2 (a, b)
\]

\[
P_{+-}(a, b) = P_{-+}(a, b) = \frac{1}{2} \sin^2 (a, b)
\]

\[
E(a, b) = P_{++} + P_{--} - P_{+-} - P_{-+}
\]

\[
E_{MQ}(a, b) = \cos^2 (a, b) = \cos^2 (a, b) - \sin^2 (a, b)
\]

\[
= \frac{1}{\sqrt{2}}
\]

for any \(\gamma\)

\[
\sin^2 \delta
\]

\[
\cos^2 \delta
\]

\[
\frac{\pi}{\delta}
\]

for \((a, b) = \frac{\pi}{\delta}\)
measure Bell inequality:

\[ S = E(a, b) - E(a, b') + E(a', b) + E(a', b') \]

with:

\[ E(a, b) = \frac{N_{++}(a, b) - N_{--}(a, b) - N_{--}(a, b) + N_{--}(a, b)}{N_{++}(a, b) + N_{--}(a, b) + N_{--}(a, b) + N_{--}(a, b)} \]

repeat for different angles between polarizer \((a, b) = \theta:\)

\[ (a, b) = (b, a') = (a', b) = \frac{\pi}{8} = 22.5 \text{ deg} \]

experimental result:

\[ 2 \sqrt{2} \]

\[ \frac{\pi}{4} \times 22.5 \text{ deg} \]

\[ 4 \times \frac{\pi}{8} = 67.5 \text{ deg} \]
Consequences of violation of Bell inequalities:

- The assumption that physical properties (e.g. Q, R, S, T) of systems have values which exist independently of observation (the Realism Assumption) is wrong.

- The assumption that experiments performed at one point in time and space (at Alices) cannot be influenced by experiments at another point in time and space (at Bobs, in a different light cone) (the Locality Assumption) is wrong.

Both of the above assumptions are sometimes called Local Realism.

Quantum mechanics violates these assumptions, as shown in experiments!

Test of Locality: The Innsbruck Experiment

Violation of Bell's Inequality under Strict Einstein Locality Conditions
G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger
[PDF (195 kB)]