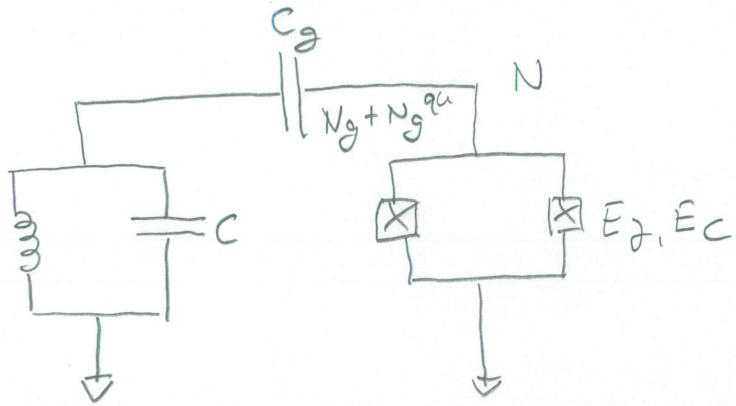
Jaynes - Cummings Hamiltonian in Circuit QED



$$\hat H = \overbrace{\frac{\hat Q^2}{2C} + \frac{\hat \Phi^2}{2L}}^{H.O.} + \frac{E_c}{2}\left(\overbrace{1 - 2(N_g + N_g^{qu})}^{=0 \text{ at } N_g = \frac{1}{2}}\right)\hat\sigma_z - \frac{E_J}{2}\hat\sigma_x$$

N_g^{qu} : quantum fluctuations of charge on capacitor C_g

$N_g = \frac{1}{2}$: consider charge degeneracy

quantum fluctuations of harmonic oscillator

$$\hat H_{HO} = \frac{1}{2} C \hat V^2 + \frac{1}{2} L \hat I^2$$

- consider only voltage fluctuations with operator $\hat V$

$$\hat V = \sqrt{\frac{\hbar\omega_r}{2C}}\,(\hat a^+ + \hat a)$$

- variance of voltage fluctuations

$$\Delta V^2 = \langle 0|\hat V^2|0\rangle - \underbrace{\langle 0|\hat V|0\rangle}_{\text{mean voltage } = 0 \text{ for } |n\rangle = |0\rangle}{}^2$$

$$= \frac{\hbar\omega_r}{2C}\underbrace{\langle 0|(\hat a^+ + \hat a)^2|0\rangle}_{=0} = \frac{\hbar\omega_r}{2C}$$

- with quantum fluctuations of charge

$$N_g^{qu} = \frac{C_g}{2e}\hat V^{qu} = \frac{C_g}{2e}\sqrt{\frac{\hbar\omega_r}{2C}}\,(\hat a^+ + \hat a)$$

Full Hamiltonian

- with change of basis $\hat{\sigma}_z \rightarrow \hat{\sigma}_x$ and $\hat{\sigma}_x \rightarrow -\hat{\sigma}_z$

$$\hat{H} = \hbar\omega_r\left(\hat{a}^+ a + \frac{1}{2}\right) + \frac{E_C}{2}\frac{C_g}{2e}\sqrt{\frac{\hbar\omega_r}{2C}}\left(\hat{a}^+ + \hat{a}\right)\hat{\sigma}_x + \frac{E_J}{2}\hat{\sigma}_z$$

- and qubit raising and lowering operators $\hat{\sigma}^+, \hat{\sigma}^-$

$\hat{\sigma}_x = (\hat{\sigma}^+ + \hat{\sigma}^-)$ we find for the interaction

$$\frac{E_c}{2}\frac{C_g}{e}\sqrt{\frac{\hbar\omega_r}{2C}}\left(\cancel{\hat{a}^+\hat{\sigma}^+}^{\rightarrow 0} + \hat{a}^+\hat{\sigma}^- + \hat{a}\hat{\sigma}^+ + \cancel{\hat{a}\hat{\sigma}^-}^{\rightarrow 0}\right)$$

rotating wave approximation (RWA)

- with $E_c = \dfrac{(2e)^2}{2C_\Sigma}$ the full Hamiltonian reads

$$\hat{H} = \hbar\omega_r\left(\hat{a}^+\hat{a}\right) + \hbar g\left(\hat{a}^+\hat{\sigma}^- + \hat{a}\,\hat{\sigma}^+\right) + \frac{E_J}{2}\hat{\sigma}_z$$

with coupling constant

$$\hbar g = \frac{C_g}{C_\Sigma}\, 2e\sqrt{\frac{\hbar\omega_r}{2C}}$$

where $\dfrac{2g}{2\pi}$ is the vacuum Rabi frequency