Estimating the Average Fidelity on a NMR Quantum Computer

Based on "Experimental Estimation of Average Fidelity of a Clifford Gate on a 7-Qubit Quantum Processor" [1]

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Outline

- Why?
  - Clifford Gates
- How (Theory)?
  - Standard Way
  - Better Way
    - Twirling protocol
- Reminder NMR
- How (Experiment)?
- Results

Why do we want to know the fidelity?

- In the construction of any gate, imperfections and noise are inevitable
- We might hence consider a physical gate as $\tilde{\mathcal{U}} = \Lambda \circ \mathcal{U}$
- The fidelity characterises the quality of a gate

$$F = \langle\psi| \Lambda(|\psi\rangle \langle\psi|) |\psi\rangle$$

What are Clifford Gates?

- The Clifford Group of operators on n qubits is defined as

$$\mathcal{C}_n = \{U \in U(2^n) | \sigma \in \pm P_n \Rightarrow U \sigma U^\dagger \in \pm P_n\}/U(1)$$

  where P_n are the (non-identity) n-qubit Pauli Matrices
- For one qubit, this just corresponds to permuting the axes on the Bloch sphere

This includes common gates such as Hadamard and Phase gates

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Why are we interested in Clifford Gates?

- For n qubits, the Clifford Group is generated by Hadamard, Phase and CNOT gates
- Many common gates are Clifford Gates, e.g. Error Correction
- Universal Quantum Computation can be achieved using only Clifford Gates and the ability to create certain states
- We thus might be interested in characterising the Fidelity of such gates

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$\rightarrow$ **because it needs too many measurements**
How many measurements does quantum process tomography need?

Remember: quantum state tomography

- Let $|\psi\rangle \in \mathcal{H}$ (Hilbert space of n-qubits)
- What's the dimension of $\mathcal{H}$?

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- What's the dimension of $\mathcal{H}$?
  $\rightarrow d := \text{dim}(\mathcal{H}) = 2^n$
- performing quantum state tomography on state $|\psi\rangle$ means estimating the density matrix $\rho = |\psi\rangle \langle\psi|$.
  Remember ρ lives in space of positive operators with trace one.
  Basis: $\{1, \sigma_x, \sigma_y, \sigma_z\}$ (dimension d^2 TRUE?)

How can we get ρ?

One qubit case

- We expand (one qubit case):
  $\rho = \frac{1}{2}(\mathrm{Tr}(\rho)1 + \mathrm{Tr}(\sigma_x\rho)\sigma_x + \mathrm{Tr}(\sigma_y\rho)\sigma_y + \mathrm{Tr}(\sigma_z\rho)\sigma_z)$
  where:
  - $\mathrm{Tr}(\rho) = 1$
  - $\mathrm{Tr}(\sigma_i\rho), i \in \{x, y, z\}$ is experimentally obtainable. Remember: $\mathrm{Tr}(\sigma_i\rho) =$ average value when performing measurement σ_i
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  Error of mean goes with $\mathcal{O}(\frac{1}{\sqrt{\#\text{ measurements}}}) \rightarrow$ if m measurements enough for precision:
  $3m = (2^2 - 1)m = \mathcal{O}(d^2 - 1)$

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Measurements needed for quantum state tomography:

$\rightarrow \mathcal{O}(d^2)$

Remember: quantum process tomography

- Question: does our process do what we want him to do?
- Situation:
  - We have a process:

$$\rho \to \mathcal{E}^{\text{have}}(\rho)$$

  - We want to be able to put the action of the process into numbers (to compare how far it is from $\rho \to \mathcal{E}^{\text{ideal}}(\rho)$).
  - A map $\rho \to \mathcal{E}(\rho)$ is determined by its action on the d^2 elements of a basis set of the set of matrices.

Number of measurements required for quantum process tomography

- We generate d^2 pure input states $|\psi_1\rangle, \ldots, |\psi_{d^2}\rangle$, whose density matrices $\rho_1 := |\psi_1\rangle\langle\psi_1|, \ldots, \rho_{d^2} := |\psi_{d^2}\rangle\langle\psi_{d^2}|$ form a basis for the matrix space of ρ.
- For all ρ_i, we determine the output of $\mathcal{E}^{have}(\rho_i)$ with quantum state tomography ($\mathcal{O}(d^2)$ measurements).

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Measurements needed for quantum process tomography:

$\rightarrow \mathcal{O}(d^4) = \mathcal{O}(2^{4n})$

- for $n = 7 \rightarrow 2^{4n} = 268435456 \approx 2.7 \times 10^8$
- has been done for at most 3 qubits (date 10th of April 2015)

What can we do then?

- We would like to benchmark a gate acting on 7 qubits
- The average fidelity of our gate is enough for us
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- We would like to benchmark a gate acting on 7 qubits
- The average fidelity of our gate is enough for us
- We don't mind if it has to consist from Clifford gates
  $\rightarrow$ we can use the twirling protocol

How does the twirling protocol work?

Some things to know:

- it works to estimate the *average fidelity* of *clifford gates*.
- A superoperator $\mathcal{U}$ is an linear operator acting on the space of linear operators.
- In our case: superoperator $\mathcal{U}$ acts on the density matrix ρ.
- $\mathcal{U}(\rho)$ is the resulting density matrix after applying gate U to the state ρ. correct???
- Superoperator we want: $\mathcal{U}$
  Superoperator we have: $\tilde{\mathcal{U}} = \Lambda \circ \mathcal{U}$,
  where Λ is the noise superoperator.
- In the end we want: a good enough estimate of the average fidelity $\bar{F}(\mathcal{U}, \tilde{\mathcal{U}})$ without making too many measurements.

Twirling mathematics 1

- The average Fidelity between $\tilde{\mathcal{U}}$ and $\mathcal{U}$:

$$\begin{aligned}\bar{F}(\mathcal{U},\tilde{\mathcal{U}}) &= \int \mathrm{d}\mu(\psi) \langle\psi|\mathcal{U}^\dagger\tilde{\mathcal{U}}(|\psi\rangle\langle\psi|)|\psi\rangle \\ &= \int \mathrm{d}\mu(\psi) \langle\psi|\mathcal{U}^\dagger \circ \Lambda \circ \mathcal{U}(|\psi\rangle\langle\psi|)|\psi\rangle \\ &= \int \mathrm{d}\mu(\psi) \langle\psi|\,\Lambda(|\psi\rangle\langle\psi|)|\psi\rangle \\ &= \bar{F}(\Lambda)\end{aligned}$$

- Equivalently: $\mathrm{d}\mu(\psi)$ (sums over states) $\leftrightarrow$ $\mathrm{d}\mu(\mathcal{V})$ (sums over random unitaries)[2].

$$\bar{F}(\Lambda) = \int \mathrm{d}\mu(\mathcal{V}) \langle\psi|\,\mathcal{V}^\dagger \circ \Lambda \circ \mathcal{V}(|\psi\rangle\langle\psi|)|\psi\rangle = \langle\psi| \int \mathrm{d}\mu(\mathcal{V})\mathcal{V}^\dagger \circ \Lambda \circ \mathcal{V}(|\psi\rangle\langle\psi|)|\psi\rangle$$

$$= F(\bar{\Lambda}). \text{ where } \bar{\Lambda} := \int \mathrm{d}\mu(\mathcal{V})\mathcal{V}^\dagger \circ \Lambda \circ \mathcal{V}$$

Twirling mathematics 2

- Trick: replace integral by sum over a finite set of unitaries $\mathcal{C}_i$ in the finite n-qubit Clifford group $\mathcal{C}_n$ [3]:
  $\bar{\Lambda} := \int \mathsf{d}\mu(\mathcal{V})\mathcal{V}^\dagger \circ \Lambda \circ \mathcal{V} \rightarrow \bar{\Lambda}_{\mathcal{C}_n} := \frac{1}{|\mathcal{C}_n|}\sum_{\mathcal{C}_i \in \mathcal{C}_n} \mathcal{C}_i^\dagger \circ \Lambda \circ \mathcal{C}_i$
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- $\bar{\Lambda}_{\mathcal{C}_n}$ is called the $\mathcal{C}_n$ twirl of Λ. The twirl $\mathcal{C}_n$ twirl of any superoperator Λ is a *depolarizing channel*.
- Thus, can be written as ($P_0 :=$ probability of no error) [4]:
$\bar{\Lambda}_{\mathcal{C}_n}(\rho) = P_0 \rho + (1 - P_0) \frac{1^{\otimes n}}{2^n}$.

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- Hence:

Average Fidelity

$$\bar{F}(\Lambda) = F(\bar{\Lambda}_{\mathcal{C}_n}) = \frac{2^n P_0 + 1}{2^n + 1}$$

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- $\rightarrow$ now we need a technique to measure P_0

Twirling mathematics 3

- $Pr(w)$ denotes the probability that a Pauli error of weight w occurs.
- $\tilde{\mathcal{U}}_C = \Lambda \circ \mathcal{U}_C$ denotes an arbitrary faulty Clifford gate
- It can be shown that [4, 5]:

$$P_0 = Pr(0) = \frac{1}{4^n}\left(1 + \frac{1}{2^n}\sum_{i=1}^{4^n-1} \mathrm{Tr}(\Lambda(\rho_i)\rho_i)\right)$$

where $\rho_i \in \mathcal{P}_n$ ($\mathcal{P}_n$ denotes the Pauli group and contains $4^n - 1$ elements.)

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Hence:

$$P_0 = Pr(0) = \frac{1}{4^n}\left(1 + \frac{1}{2^n}\sum_{i=1}^{4^n-1} \mathrm{Tr}(\mathcal{U}_C(\rho_i)\tilde{\mathcal{U}}_C(\rho_i))\right)$$

So how many measurements do we need?

- At most $4^n - 1 = \mathcal{O}(d^2)$, for $n = 7$, $4^n - 1 = 16384$
- Denote:
  - $Prob(\epsilon)$: probability of event $\epsilon : |\bar{x} - \mu| > \delta$
  - m: number of measurements
- Hoeffding's inequality:

$$Prob(\epsilon) \leq 2e^{-2\delta^2 m}$$

- $\Rightarrow$ number of needed measurements m:

$$m \leq \frac{\ln(2/Prob(\epsilon))}{2\delta^2}$$

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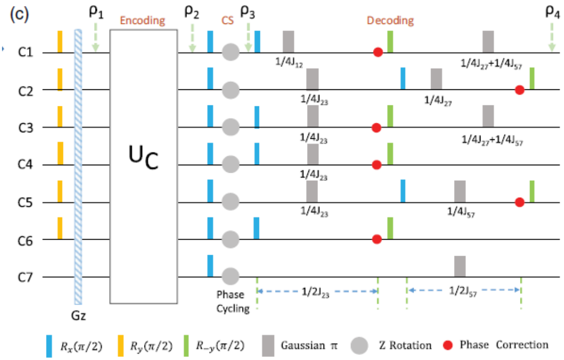
For given $Prob(\epsilon)$ and δ:

Number of measurements needed is independent of number of qubits!

For $\epsilon = 1\%$ and $\delta = 0.04 \Rightarrow m \leq 1656$.
Remember: full quantum process tomography: $m \approx 2.7 \times 10^6$

How do we perform the experiments?

The gate $\mathcal{U}$ of interest we choose evolves $\rho_1 = ZI^{\otimes 6}$ to $\rho_2 = Z^{\otimes 7}$



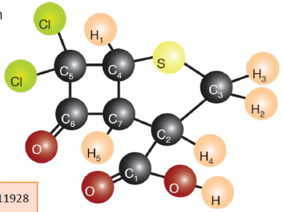
where $\rho_3 = |0\rangle\langle 0|^{\otimes 7} + |1\rangle\langle 1|^{\otimes 7}$ and $\rho_4 = |0\rangle\langle 0|^{\otimes 6} \otimes Z_7$

Molecule we use

C-13 labeled 12-qubit system

Dichloro–cyclobutanone

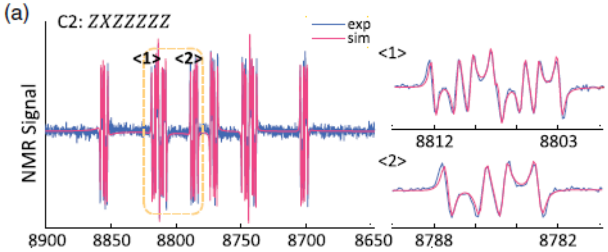
	C1	C2	C3	C4	C5	C6	C7	H1	H2	H3	H4	H5
C1	30020											
C2	57.58	8779										
C3	-2.00	32.70	6245									
C4	0	0.30	0	10333								
C5	1.25	2.62	-1.11	33.16	15745							
C6	5.54	-1.66	0	-3.53	33.16	34381						
C7	-1.25	37.48	0.94	29.02	21.75	34.57	11928					
H1	0	0	2.36	166.6	4.06	5.39	8.61	3310				
H2	4.41	1.86	146.6	2.37	0	0	0	0	2468			
H3	1.81	3.71	146.6	2.37	0	0	0	0.18	-12.41	2158		
H4	-13.19	133.6	-6.97	6.23	0	5.39	3.78	-0.68	1.28	6.00	2692	
H5	7.87	-8.35	3.35	8.13	2.36	8.52	148.5	8.46	-1.06	-0.36	1.30	3649
T1	8.015	3.611	1.834	3.722	12.95	8.157	3.636	3.831	2.128	2.278	2.654	3.472
T2	1.611	0.877	1.122	0.792	1.143	1.912	0.531	0.337	N/A	N/A	0.318	0.276



How do we perform the experiments?

We are interested in isolating just the error caused by the gate

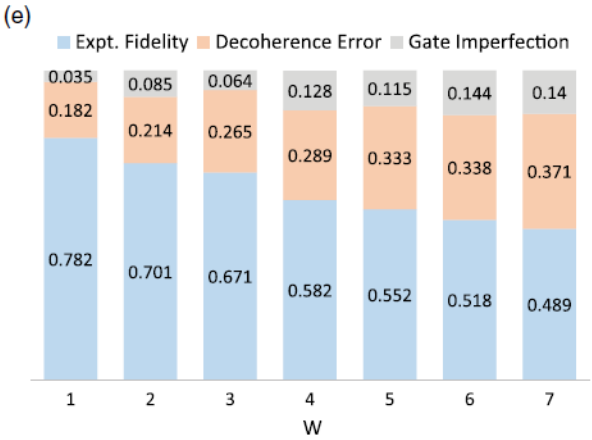
- After creating the state, we calibrate it by comparing its nmr signal to that of the equilibrium state
- Error due to decoherence was calculated theoretically and removed from the measured result



Results

	Number of Expt.		Calibration		Average Fidelity			
Weight	k	k_w	t (ms)	F_i	F_e	E_d	$F_{d,c}$
w=1	21	3	2	0.977±0.024	0.782±0.012	18.9%	0.965
w=2	189	22	26	0.915±0.029	0.701±0.047	23.4%	0.915
w=3	945	101	34	0.895±0.039	0.671±0.042	28.3%	0.936
w=4	2835	272	49	0.866±0.025	0.582±0.033	33.2%	0.872
w=5	5103	505	53	0.838±0.041	0.552±0.033	37.6%	0.885
w=6	5103	524	55	0.861±0.030	0.518±0.026	39.5%	0.856
w=7	2187	229	60	0.865±0.031	0.489±0.029	43.1%	0.860
Total	16383	1656	N/A	0.858	0.547	37.4%	0.874

Results



(e)
Expt. Fidelity
Decoherence Error
Gate Imperfection
0.035
0.182
0.782
0.085
0.214
0.701
0.064
0.265
0.671
0.128
0.289
0.582
0.115
0.333
0.552
0.144
0.338
0.518
0.14
0.371
0.489
1
2
3
4
5
6
7
W

Results

$$\bar{F}(\Lambda) = \frac{2^n P_0 + 1}{2^n + 1}$$

Measured value of $P_0 = 87.4\%$ gives fidelity $\bar{F} = 87.5\%$.

- Decomposing the gate into 18 1- and 2-qubit gates gives average fidelity of 99% or 96% before accounting for decoherence
- The best previous nmr result on 3-qubit gates was 99% (86%)
- Xmon 2-qubit gates have shown 99.0-99.4% fidelities
- This is the only such gate characterisation for more than 3-qubits

Summary

- Clifford gates were introduced and their importance was highlighted
- We showed how to estimate the average fidelity of Clifford gates using a number of measurements which is constant in the number of qubits by employing the twirling protocol.
- We described the measurement setup for measuring the average fidelity of a 7-qubit NMR system.
- Finally we analysed the results and found that they compared favourably to other experiments

Thank you for your attention!

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Questions?
or
Comments?

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