2.5 Dynamics of Quantum Systems

2.5.1 The Schrödinger equation

QM postulate: The time evolution of a state $|\psi\rangle$ of a closed quantum system is described by the **Schrödinger equation**

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

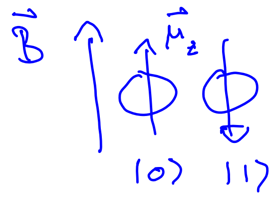
where H is the hermitian operator known as the **Hamiltonian** describing the closed system.

Reminder: A **closed quantum system** is one which does not interact with any other system.

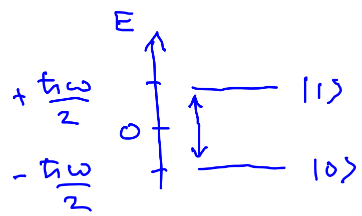
general solution for a time independent Hamiltonian H:

$$|\psi(t)\rangle = \exp\left[\frac{-iHt}{\hbar}\right]|\psi(0)\rangle$$

example: e.g. electron spin in a field



energy level diagram:



Hamiltonian for spin 1/2 in a magnetic field: $H = -\frac{\hbar\omega}{2} Z$

$$H = -\frac{\hbar\omega}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)$$

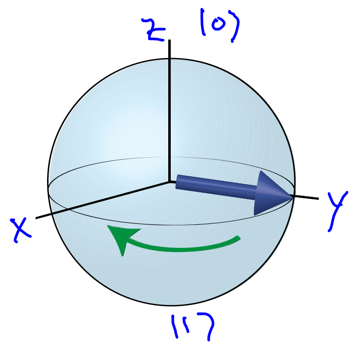
$$|\psi(0)\rangle = |0\rangle \longrightarrow |\psi(t)\rangle = e^{\frac{i\omega}{2}t}|0\rangle$$

$$|\psi(0)\rangle = |1\rangle \longrightarrow |\psi(t)\rangle = e^{-\frac{i\omega}{2}t}|1\rangle$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\cancel{e^{i\frac{\omega}{2}t}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right)$$

interpretation of dynamics on the Bloch sphere:



$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

$$\Longrightarrow \quad \theta = \frac{\pi}{2}, \quad \varphi = -\omega t$$

this is a rotation around the equator of the Bloch sphere with **Larmor precession frequency** ω

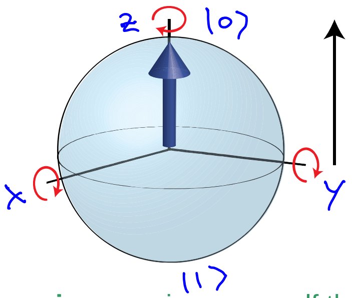
2.5.2 Rotation of qubit state vectors and rotation operators

when exponentiated the Pauli matrices give rise to rotation matrices around the three orthogonal axis in 3-dimensional space.

$$R_x(\theta) = e^{-i\theta X/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} X = \begin{pmatrix} \cos\frac{\theta}{2} & -i \sin\frac{\theta}{2} \\ -i \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_y(\theta) = e^{-i\theta Y/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

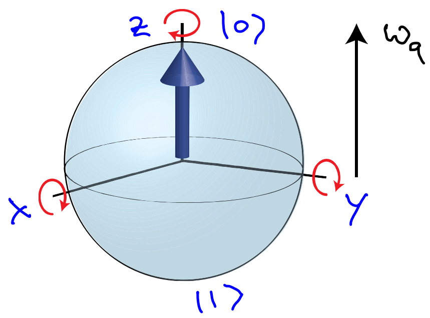
$$R_z(\theta) = e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$



If the Pauli matrices **X**, **Y** or **Z** are present in the Hamiltonian of a system they will give rise to rotations of the qubit state vector around the respective axis.

exercise: convince yourself that the operators $\mathbf{R}_{x,y,z}$ do perform rotations on the qubit state written in the Bloch sphere representation.

2.5.3 Preparation of specific qubit states



initial state **|0>:**

prepare excited state by rotating around **x** or **y** axis:

X$_\pi$ pulse: $\Omega_x t = \pi$; $|0\rangle - \boxed{X_\pi} - |1\rangle$

Y$_\pi$ pulse: $\Omega_y t = \pi$; $|0\rangle - \boxed{Y_\pi} - -i|1\rangle$

preparation of a superposition state:

X$_{\pi/2}$ pulse: $\Omega_x t = \frac{\pi}{2}$ $\quad |0\rangle - \boxed{X_{\pi/2}} - \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

Y$_{\pi/2}$ pulse: $\Omega_y t = \frac{\pi}{2}$ $\quad |0\rangle - \boxed{Y_{\pi/2}} - \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$

in fact such a pulse of chosen length and phase can prepare any single qubit state, i.e. any point on the Bloch sphere can be reached

2.6 Quantum Measurement

Quantum measurement is done by having a closed quantum system interact in a controlled way with an external system from which the state of the quantum system under measurement can be recovered.

- example to be discussed: dispersive measurement in cavity QED

2.6.1 The quantum measurement postulate

QM postulate: **quantum measurement** is described by a set of operators **$\{M_m\}$** acting on the state space of the system. The **probability p of a measurement result m** occurring when the state ψ is measured is

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$$

the **state of the system after the measurement** is

$$|\psi'\rangle = \frac{M_m |\psi\rangle}{\sqrt{p(m)}}$$

completeness: the sum over all measurement outcomes has to be unity

$$1 = \sum_m p(m) = \sum_m \langle \psi | M_m^\dagger M_m | \psi \rangle$$

2.6.2 Example: projective measurement of a qubit in state ψ in its computational basis

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

measurement operators:

$$|M_0\rangle = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \quad M_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

measurement probabilities:

$$p(0) = \langle \psi | M_0^\dagger M_0 | \psi \rangle = \alpha^* \alpha \langle 0 | 0 \rangle = |\alpha|^2$$

$$p(1) = \langle \psi | M_1^\dagger M_1 | \psi \rangle = \beta^* \beta \langle 1 | 1 \rangle = |\beta|^2$$

state after measurement:

$$\frac{M_0 |\psi\rangle}{\sqrt{p(0)}} = \frac{\alpha |0\rangle}{\sqrt{|\alpha|^2}} = \frac{\alpha}{|\alpha|} |0\rangle$$

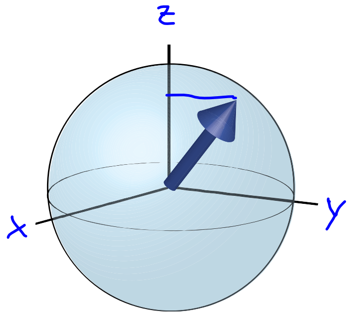
$$\frac{M_1 |\psi\rangle}{\sqrt{p(1)}} = \frac{\beta |1\rangle}{\sqrt{|\beta|^2}} = \frac{\alpha}{|\alpha|} |1\rangle$$

measuring the state again after a first measurement yields the same state as the initial measurement with unit probability

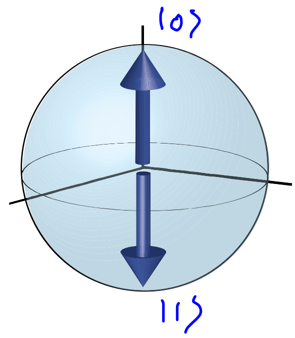
2.6.3 Interpretation of the Action of a Projective Measurement

One way to determine the state of a qubit is to measure the projection of its state vector along a given axis, say the z-axis.

On the Bloch sphere this corresponds to the following operation:



After a projective measurement is completed the qubit will be in either one of its computational basis states.

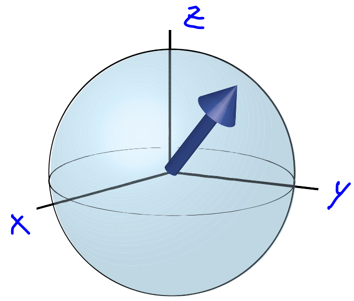


In a repeated measurement the projected state will be measured with certainty.

Information content in a single qubit state

- infinite number of qubit states
- but single measurement reveals only 0 or 1 with probabilities $|\alpha|^2$ or $|\beta|^2$
- measurement will collapse state vector on basis state
- to determine α and β an infinite number of measurements has to be made

But if not measured the qubit contains 'hidden' information about α and β.



2.7 Multiple Qubits

2.7.1 Two Qubits

2 classical bits with states:

bit 1	bit 2
0	0
0	1
1	0
1	1

- 2^n different states (here n=2)
- but only one is realized at any given time

2 qubits with quantum states:

qubit 1 qubit 2

$|00\rangle$
$|01\rangle$
$|10\rangle$
$|11\rangle$

- 2^n basis states (n=2)
- can be realized simultaneously
- quantum parallelism

2^n complex coefficients describe quantum state

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

normalization condition

$$\sum_{ij} |\alpha_{ij}|^2 = 1$$

2.7.2 Composite quantum systems

QM postulate: The state space of a composite systems is the tensor product of the state spaces of the component physical systems. If the component systems have states ψ_i the composite system state is

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$$

This is a product state of the individual systems.

example:

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$
$$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

$$\rightarrow \quad |\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\psi_2\rangle$$
$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

exercise: Write down the state vector (matrix representation) of two qubits, i.e. the tensor product, in the computational basis. Write down the basis vectors of the composite system.

there is no generalization of Bloch sphere picture to many qubits

2.7.3 Information content in multiple qubits

- 2^n complex coefficients describe the state of a composite quantum system with n qubits
- Imagine to have 500 qubits, then 2^{500} complex coefficients describe their state.
- How to store this state?
  - 2^{500} is larger than the number of atoms in the universe.
  - It is impossible in classical bits.
  - This is also why it is hard to simulate quantum systems on classical computers.
- A quantum computer would be much more efficient than a classical computer at simulating quantum systems.
- Make use of the information that can be stored in qubits for quantum information processing!

2.7.4 Entanglement

Definition: An **entangled state** of a composite system is a state that cannot be written as a product state of the component systems.

example: an entangled 2-qubit state (one of the Bell states)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

What is special about this state? Try to write it as a product state!

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle \; ; \; |\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

$$|\psi_1\psi_2\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

$$|\psi\rangle \overset{!}{=} |\psi_1\psi_2\rangle \Rightarrow \alpha_1\alpha_2 = \frac{1}{\sqrt{2}} \wedge \beta_1\beta_2 = \frac{1}{\sqrt{2}} \Rightarrow \alpha_1\beta_2 \neq 0 \wedge \alpha_2\beta_1 \neq 0 \, !$$

It is not possible! This state is special, it is entangled!

Use this property as a resource in quantum information processing:
- super dense coding
- teleportation
- error correction

2.7.5 Measurement of a single qubit in an entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

measurement of first qubit:

$$p_1(0) = \langle\psi|(M_0 \otimes I)^\dagger (M_0 \otimes I)|\psi\rangle = \frac{1}{\sqrt{2}}\langle 00| \frac{1}{\sqrt{2}}|00\rangle = \frac{1}{2}$$

post measurement state:

$$|\psi'\rangle = \frac{(M_0 \otimes I)|\psi\rangle}{\sqrt{p_1(0)}} = \frac{\frac{1}{\sqrt{2}}|00\rangle}{\frac{1}{\sqrt{2}}} = |00\rangle$$

measurement of qubit two will then result with certainty in the same result:

$$p_2(0) = \langle\psi'|(I \otimes M_0)^\dagger (I \otimes M_0)|\psi'\rangle = 1$$

The two measurement results are **correlated**!

- Correlations in quantum systems can be stronger than correlations in classical systems.
- This can be generally proven using the **Bell inequalities** which will be discussed later.
- Make use of such correlations as a **resource** for information processing
  - super dense coding, teleportation, error corrections