2.7.6 Super Dense Coding

task: Try to transmit two bits of classical information between Alice (A) and Bob (B) using only one qubit. 
  ○ As Alice and Bob are living in a quantum world they are allowed to use one pair of entangled qubits that they have prepared ahead of time.

protocol:
A) Alice and Bob each have one qubit of an entangled pair in their possession
\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right)
\]
B) Alice does a quantum operation on her qubit depending on which 2 classical bits she wants to communicate
C) Alice sends her qubit to Bob
D) Bob does one measurement on the entangled pair

```
| shared entanglement | \[ \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \] |
| local operations    | \[ X, Y, i, z, i, \bar{X}, \bar{Y} \] |
| send Alice's qubit to Bob | |
| Bob measures |
```

<table>
<thead>
<tr>
<th>bits to be transferred</th>
<th>Alice's operation</th>
<th>resulting 2-qubit state</th>
<th>Bob's measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>I</td>
<td>[ I, \psi \rangle = \frac{1}{\sqrt{2}} (</td>
<td>00\rangle +</td>
</tr>
<tr>
<td>01</td>
<td>Z</td>
<td>[ Z, \psi \rangle = \frac{1}{\sqrt{2}} (</td>
<td>00\rangle -</td>
</tr>
<tr>
<td>10</td>
<td>X</td>
<td>[ X, \psi \rangle = \frac{1}{\sqrt{2}} (</td>
<td>10\rangle +</td>
</tr>
<tr>
<td>11</td>
<td>\bar{Y}</td>
<td>[ \bar{Y}, \psi \rangle = \frac{1}{\sqrt{2}} (</td>
<td>11\rangle -</td>
</tr>
</tbody>
</table>

comments:
- two qubits are involved in protocol BUT Alice only interacts with one and sends only one along her quantum communications channel
- two bits cannot be communicated sending a single classical bit along a classical communications channel

2.7.7 Experimental demonstration of super dense coding using photons

Generating polarization entangled photon pairs using **Parametric Down Conversion**:

- 1 UV-photon $\rightarrow$ 2 "red" photons
- conservation of energy $\omega_p = \omega_s + \omega_i$
- momentum $\vec{k}_p = \vec{k}_s + \vec{k}_i$
- Polarisation correlation (typ II)

optically nonlinear medium: BBO (BaB$_2$O$_4$) beta barium borate

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|H\rangle|V\rangle - |V\rangle|H\rangle)$$

**Bell state measurement**

$$\Psi^- = \frac{1}{\sqrt{2}} (|HH\rangle - |VV\rangle)$$
$$\Psi^+ = \frac{1}{\sqrt{2}} (|HH\rangle + |VV\rangle)$$
$$\Phi^+ = \frac{1}{\sqrt{2}} (|HH\rangle + |VV\rangle)$$
$$\Phi^- = \frac{1}{\sqrt{2}} (|HH\rangle - |VV\rangle)$$

$H =$ horizontal polarization
$V =$ vertical polarization

2.8 Two Qubit Quantum Logic Gates

2.8.1 The controlled NOT gate (CNOT)

function:

\[
\begin{align*}
|00\rangle & \rightarrow |00\rangle \\
|01\rangle & \rightarrow |01\rangle \\
|10\rangle & \rightarrow |11\rangle \\
|11\rangle & \rightarrow |10\rangle
\end{align*}
\]

\[|A,B\rangle \rightarrow |A,A\otimes B\rangle \]

addition mod 2 of basis states

CNOT circuit:

control qubit

target qubit

comparison with classical gates:
- XOR is not reversible
- CNOT is reversible (unitary)

Universality of controlled NOT:
Any multi qubit logic gate can be composed of CNOT gates and single qubit gates X,Y,Z.

2.8.2 Application of CNOT: generation of entangled states (Bell states)

\[
\begin{align*}
|00\rangle & \xrightarrow{H_1} \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\
|01\rangle & \xrightarrow{H_1} \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \\
|10\rangle & \xrightarrow{H_1} \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle) \\
|11\rangle & \xrightarrow{H_1} \frac{1}{\sqrt{2}} (|11\rangle - |00\rangle)
\end{align*}
\]

\[
\begin{align*}
|00\rangle & \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\
|01\rangle & \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \\
|10\rangle & \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle) \\
|11\rangle & \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}} (|11\rangle - |00\rangle)
\end{align*}
\]

exercise: Write down the unitary matrix representations of the CNOT in the computational basis with qubit 1 being the control qubit. Write down the matrix in the same basis with qubit 2 being the control bit.
2.8.3 Implementation of CNOT using the Ising interaction

Ising interaction:
\[ H = - \sum_{i,j} J_{ij} \hat{S}_i \cdot \hat{S}_j \]
 pair wise spin interaction

generic two-qubit interaction:
\[ H = - J \hat{S}_1 \cdot \hat{S}_2 \]

J > 0: ferromagnetic coupling \quad J < 0: anti-ferrom. coupling

\[ E \uparrow + J \quad 1\uparrow \downarrow \text{ or } 1\uparrow 1\uparrow \]
\[ -J \quad 1\uparrow \downarrow \text{ or } 1\downarrow 1\uparrow \]

2-qubit unitary evolution:
\[ C(\theta) = e^{-i \frac{\theta}{2} \hat{S}_1 \cdot \hat{S}_2} \]

BUT this does not realize a CNOT gate yet. Additionally, single qubit operations on each of the qubits are required to realize a CNOT gate.

CNOT realization with the Ising-type interaction

CNOT - unitary:
\[ C_{\text{ROT}} = e^{-i \frac{3\pi}{4} R_{z} \left( \frac{\pi}{4} \right) C \left( \frac{3\pi}{4} \right) R_{z} \left( \frac{\pi}{4} \right) R_{x} \left( \frac{\pi}{2} \right) R_{z} \left( \frac{\pi}{4} \right) R_{z} \left( \frac{\pi}{4} \right) C \left( \frac{3\pi}{4} \right) } \]

circuit representation:

Any physical two-qubit interaction that can produce entanglement can be turned into a universal two-qubit gate (such as the CNOT gate) when it is augmented by arbitrary single qubit operations.

2.9 Quantum Teleportation

**Task**: Alice wants to transfer an unknown quantum state $\psi$ to Bob only using a **one entangled pair** of qubits and **classical information** as a resource.

**note**:
- Alice does not know the state to be transmitted
- Even if she knew it the classical amount of information that she would need to send would be infinite.

The **teleportation circuit**:

![Teleportation Circuit Diagram]


**2.9.1 How does it work?**

1. $|\psi\rangle \otimes \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \left( |\psi_0\rangle + |\psi_1\rangle \right)

2. CNOT between qubit to be teleported and one bit of the entangled pair:

3. Hadamard on qubit to be teleported:

4. Measurement of qubit 1 and 2, classical information transfer and single bit manipulation on target qubit 3:

5. Quantum state is transferred.
2.9.2 (One) Experimental Realization of Teleportation using Photon Polarization:

- parametric down conversion (PDC) source of entangled photons
- qubits are polarization encoded


**Experimental Implementation**

start with states

\[
|\psi_1\rangle = \alpha |H\rangle + \beta |V\rangle
\]

\[
|\psi_{23}\rangle = \frac{1}{\sqrt{2}} \left( |HV\rangle - |VH\rangle \right)
\]

combine photon to be teleported (1) and one photon of entangled pair (2) on a 50/50 beam splitter (BS) and measure (at Alice) resulting state in Bell basis.

analyze resulting teleported state of photon (3) using polarizing beam splitters (PBS) single photon detectors

- polarizing beam splitters (PBS) as detectors of teleported states
teleportation papers for you to present:

**Experimental Realization of Teleporting an Unknown Pure Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels**

D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu

**Unconditional Quantum Teleportation**

A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik
Abstract » Full Text » PDF »

**Complete quantum teleportation using nuclear magnetic resonance**

M. A. Nielsen, E. Knill, R. Laflamme
Nature 396, 52 - 55 (05 Nov 1998) Letters to Editor
Abstract | Full Text | PDF | Rights and permissions | Save this link

**Deterministic quantum teleportation of atomic qubits**

Nature 429, 737 - 739 (17 Jun 2004) Letters to Editor
Abstract | Full Text | PDF | Rights and permissions | Save this link

**Deterministic quantum teleportation with atoms**

Nature 429, 734 - 737 (17 Jun 2004) Letters to Editor
Abstract | Full Text | PDF | Rights and permissions | Save this link

**Quantum teleportation between light and matter**

Jacob F. Sherson, Hanna Krauter, Rasmus K. Olsson, Brian Julsgaard, Klemens Hammerer, Ignacio Cirac, Eugene S. Polzik
Nature 443, 557 - 560 (05 Oct 2006) Letters to Editor
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