Phase particle in a potential well

$$U(\delta) = \frac{I_0 \phi_0}{2\pi} \left( -\frac{I_0}{I_c} \delta - \cos \delta \right)$$

cosine potential for $I_0 = 0$:

'tilted washboard' potential for $I_0 \neq 0$:

potential barrier:

$$U_o = 2E_\delta \left[ \sqrt{1 - \delta^2} - \delta \arccos \delta \right]$$

oscillation frequency:

$$\omega_o = \omega_0 \left( 1 - \delta^2 \right)^{1/4} = \sqrt{\frac{U_o}{m}}$$

with: $\delta = I_0/I_c \quad \omega_0 = \sqrt{\frac{2\pi I_c}{\phi_0}}$

Current-voltage characteristics

typical I-V curve of underdamped Josephson junctions:
Thermal Activation and Quantum Tunneling:
thermal activation rate:

\[ \Gamma_{th} = a_t \frac{U_0}{2\pi k_B T} \exp\left(-\frac{U_0}{k_B T}\right) \]
damping dependent prefactor
quantum tunneling rate:

\[ \Gamma_q = a_q \frac{U_0}{2\pi} \exp\left(-\frac{3}{5} \frac{U_0}{U_0} \right) \]
calculated using WKB method (exercise)

\[ \Gamma_q = a_q U_0 \exp\left(-\frac{1}{8} \frac{1}{\pi} \sqrt{2m(U_0^2 - \epsilon)}\right) \]
energy level quantization:

\[ E_n = \pm \omega_0 \left(n + \frac{1}{2}\right) \]
neglecting non-linearity

Quantum Mechanics of a Macroscopic Variable: The Phase Difference of a Josephson Junction
JOHN CLARKE, ANDREW N. CLELAND, MICHEL H. DEVORET, DANIEL ESTEVE, and JOHN M. MARTINIS

Macroscopic quantum effects in the current-biased Josephson junction
M. H. Devoret, D. Esteve, C. Urbina, J. Martinis, A. Cleland, J. Clarke
in Quantum tunneling in condensed media, North-Holland (1992)

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Early Results (1980’s)

search for macroscopic quantum effects in superconducting circuits

theoretical predictions:
- tunneling ✔
- energy level quantization ✔
- coherence ❌

A.J. Leggett et al.,
Prog. Theor. Phys. Suppl. 69, 80 (1980),

short coherence times due to strong coupling to em environment

The Current Biased Phase Qubit

operating a current biased Josephson junction as a superconducting qubit:

initialization:
wait for $|1\rangle$ to decay to $|0\rangle$, e.g. by spontaneous emission at rate $\gamma_{10}$

Read-Out Ideas

measuring the state of a current biased phase qubit

tunneling:
- prepare state $|1\rangle$ (pump)
- wait ($\Gamma_1 \sim 10^3 \Gamma_0$)
- detect voltage
- $|1\rangle = \text{voltage}, |0\rangle = \text{no voltage}$

pump and probe pulses:
- prepare state $|1\rangle$ (pump)
- drive $\omega_{21}$ transition (probe)
- observe tunneling out of $|2\rangle$

 tipping pulse:
- prepare state $|1\rangle$
- apply current pulse to suppress $U_0$
- observe tunneling out of $|1\rangle$
A Charge Qubit: The Cooper Pair Box

\[ H = 4E_C n^2 \]
\[ H = 4E_C (n - n_g)^2 - E_J \cos \delta \]
\[ [\delta, n] = i \quad \rightarrow \quad e^{\pm i\delta} |n\rangle = |n \pm 1\rangle \]

\[ H = \sum_n \left[ 4E_C(n - n_g)^2|n\rangle\langle n| - \frac{E_J}{2} (|n\rangle\langle n+1| + |n+1\rangle\langle n|) \right] \]

Charging energy: \[ E_C = \frac{e^2}{2(C_g + C_J)} \]

Gate charge: \[ n_g = \frac{C_g V_g}{2e} \]

Josephson energy: \[ E_J = \frac{I_0 \Phi_0}{2\pi} = \frac{\hbar \Delta}{8e^2 R_I} \]

Cooper pair box Hamiltonian:

Hamiltonian: \[ \hat{H} = E_c (\hat{N} - n_g)^2 - E_J \cos \hat{\delta} = \frac{E_J}{2} (e^{\hat{\delta}} + e^{-\hat{\delta}}) \]

electrostatic
\[ \text{magnetic energy} \]
charging energy
Josephson coupling energy

\[ E_c = \frac{(2e)^2}{2 C} \quad E_J = \frac{\Phi_0}{\pi e} \]

Hamiltonian in charge representation:

\[ \hat{H} = E_c (\hat{N} - n_g)^2 |N\rangle\langle N| - \frac{E_J}{2} \sum_N (|N+1\rangle\langle N| + |N\rangle\langle N+1|) \]

easy to diagonalize numerically

relation between phase and number basis:

\[ |\delta\rangle = \frac{1}{\sqrt{2\pi}} \sum_N e^{i\delta N} |N\rangle \quad \text{with} \quad e^{i\delta} |N\rangle = |N+1\rangle \]
Phase representation of Cooper pair box Hamiltonian:

\[
\hat{H} = E_c (\hat{N} - N_g)^2 - E_J \cos \hat{\phi} \\
= E_c (-i \frac{\partial}{\partial \phi} - N_g)^2 - E_J \cos \hat{\phi}
\]

with

\[
\hat{\phi} = \frac{\hat{n}}{2e} - \frac{1}{2e} \frac{\partial}{\partial \phi}
\]

Equivalent solution to the Hamiltonian can be found in both representations, e.g. by numerically solving the Schrödinger equation for the charge \((N)\) representation or analytically solving the Schrödinger equation for the phase \((\phi)\) representation.

\[
\hat{H} |\psi\rangle = E |\psi\rangle
\]

Energy Levels

- **Energy level diagram for** \(E_J = 0\):
  - energy bands are formed
  - bands are periodic in \(N_g\)

- **Energy bands for finite** \(E_J\):
  - Josephson coupling lifts degeneracy
  - \(E_J\) scales level separation at charge degeneracy

- **Tunable artificial atom**
  - \(E_J \gg k_B T\)
Charge and Phase Wave Functions ($E_j << E_c$)

Charge and Phase Wave Functions ($E_j \sim E_c$)
**Tuning the Josephson Energy**

split Cooper pair box in perpendicular field

\[ H = E_C (N - N_g)^2 - E_J \max \cos \left( \frac{\Phi_{\text{ext}}}{\Phi_0} \right) \]

SQUID modulation of Josephson energy

\[ E_J = E_J \max \cos \left( \pi \frac{\Phi_{\text{ext}}}{\Phi_0} \right) \]

consider two state approximation

\[ \sum_N \left[ E_C (N - N_g)^2 |N\rangle \langle N| - \frac{E_J}{2} (|N\rangle \langle N + 1| + |N + 1\rangle \langle N|) \right] \]

Restricting to a two-charge Hilbert space:

\[ N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1 - \sigma_z}{2} \]

\[ \cos \delta = \frac{\sigma_z}{2} \]

\[ H_{\text{CPB}} = -\frac{E_C}{2} (1 - 2N_g)\sigma_z - \frac{E_J}{2} \sigma_x \]

\[ = -\frac{1}{2} (E_c \sigma_z + E_J \sigma_x) \]

**Two State Approximation**


Cavity QED with Electronic Circuits

Cavity Quantum Electrodynamics

coupling photons to qubits:

Jaynes-Cummings Hamiltonian

\[ H = \hbar \omega_r \left( a^\dagger a + \frac{1}{2} \right) + \frac{\hbar \omega_a}{2} \sigma^z + \hbar g (a^\dagger \sigma^- + a \sigma^+) + H_\kappa + H_\gamma \]

strong coupling limit \( g = \frac{dE_0}{\hbar} > \gamma, \kappa, 1/t_{\text{transit}} \)

D. Walls, G. Milburn, Quantum Optics (Springer-Verlag, Berlin, 1994)
**Dressed States Energy Level Diagram**

\[ H = \hbar \omega_r \left( a^\dagger a + \frac{1}{2} \right) + \frac{\hbar \omega_a}{2} \sigma^z + \hbar g (a^\dagger \sigma^- + a \sigma^+) \]

- in resonance:
  \[ \omega_a - \omega_r = \Delta = 0 \]

- strong coupling limit:
  \[ g = \frac{dE_0}{\hbar} > \gamma, \kappa \]

**Jaynes-Cummings Ladder**

Atomic cavity quantum electrodynamics reviews:

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**Cavity Quantum Electrodynamics (QED)**

- alkali atoms
  - MPQ, Caltech, ...

- Rydberg atoms
  - ENS, MPQ, ...

- superconductor circuits
  - Yale, Delft, NTT, ETHZ, NIST, ...

- semiconductor quantum dots
  - Wurzburg, ETHZ, Stanford, ...
**Vacuum Rabi Oscillations with Rydberg Atoms**

Review: J. M. Raimond, M. Brune, and S. Haroche  
P. Hyafil, ..., J. M. Raimond, and S. Haroche,  

**Vacuum Rabi Mode Splitting with Alkali Atoms**

R. J. Thompson, G. Rempe, & H. J. Kimble,  
A. Boca, ..., J. McKeever, & H. J. Kimble  
Cavity QED with Superconducting Circuits

... in superconducting circuits:

Circuit quantum electrodynamics

coherent quantum mechanics with individual photons and qubits ...


Circuit Quantum Electrodynamics

elements

- the cavity: a superconducting 1D transmission line resonator with large vacuum field $E_0$ and long photon lifetime $1/\kappa$
- the artificial atom: a Cooper pair box with large $E_J/E_C$ with large dipole moment $d$ and long coherence time $1/\gamma$

A. Blais et al., PRA 69, 062320 (2004)
Vacuum Field in 1D Cavity

Voltage across resonator in vacuum state ($n = 0$)

$$V_{0,\text{rms}} = \sqrt{\frac{\hbar \nu_t}{2C}} \approx 1 \mu V$$

Harmonic oscillator

$$H_r = \hbar \nu_t \left( a^+ a + \frac{1}{2} \right)$$

$$E_0 = \frac{V_{0,\text{rms}}}{b} \approx 0.2 V/m$$

$\times 10^8$ larger than $E_0$ in 3D microwave cavity

Resonator Quality Factor and Photon Lifetime

Resonance frequency:

$$\nu_r = 6.04 \text{ GHz}$$

Quality factor:

$$Q = \frac{\nu_r}{\delta \nu_r} \approx 10^4$$

Photon decay rate:

$$\frac{\kappa}{2\pi} = \frac{\nu_r}{Q} \approx 0.8 \text{ MHz}$$

Photon lifetime:

$$T_r = \frac{1}{\kappa} \approx 200 \text{ ns}$$
**Qubit/Photon Coupling in a Circuit**

![Qubit coupled to resonator diagram]

Coupling strength:
\[
\hbar g = e V_{0,\text{rms}} \frac{C_g}{C_{\gamma^2}}
\]

\[
\nu_{\text{vac}} = \frac{g}{\pi} \approx 1 \ldots 300 \text{ MHz}
\]

\(g \gg [\kappa, \gamma]\) possible!

Large effective dipole moment:
\[
d = \frac{\hbar g}{E_0} \sim 10^2 \ldots 10^4 e a_0
\]

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**Circuit QED with One Photon**

![Circuit QED with one photon diagram]

Superconducting cavity QED circuit

Resonant Vacuum Rabi Mode Splitting...

... with one photon \((n = 1)\):

very strong coupling:

\[ g_{ge}/\pi = 308 \text{ MHz} \]

\[ \kappa, \gamma < 1 \text{ MHz} \]

\[ g_{ge} \gg \kappa, \gamma \]

forming a 'molecule' of a qubit and a photon

\[ |1\pm\rangle = (|g, 1\rangle \pm |e, 0\rangle)/\sqrt{2} \]

this data: J. Fink et al., Nature (London) 454, 315 (2008)

How to Measure Single Microwave Photons

- average power to be detected

\[ \langle n = 1 \rangle h\omega, \kappa/2 \approx P_{RF} = -140 \text{ dBm} = 10^{-17} \text{ W} \]

- efficient with cryogenic low noise HEMT amplifier \((T_N = 6 \text{ K})\)

- prevent leakage of thermal photons (cold attenuators and circulators)
Measurement Setup

- microwave electronics
- 20 mK cryostat
- sample mount
- cold stage