Quantum Voltage fluctuations of harmonic oscillator:

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} = \frac{1}{2} C \hat{V}^2 + \frac{\hat{\Phi}^2}{2L}$$

voltage operator: $\hat{V} = \sqrt{\frac{\hbar\omega_r}{2C}} (\hat{a}^\dagger + \hat{a})$ 'position'

$\hat{\Phi} = \sqrt{\frac{\hbar\omega_r}{2L}} (\hat{a}^\dagger - \hat{a})$ 'momentum'

voltage across oscillator in the ground state:

$\langle 0 | \hat{V} | 0 \rangle = 0$ since $\langle 0 | \hat{a} | 0 \rangle = \langle 0 | \hat{a}^\dagger | 0 \rangle = 0$

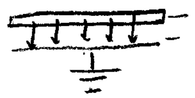
$$\Delta V_0^2 = \langle \hat{V}^2 \rangle_0 - \langle \hat{V} \rangle_0^2 = \langle \hat{V}^2 \rangle_0 =$$

$$= \langle 0 | \hat{V}^2 | 0 \rangle = \frac{\hbar\omega_r}{2C} \underbrace{\langle 0 | a^\dagger a^\dagger + a^\dagger a + a a^\dagger + a a | 0 \rangle}_{1}$$

$$\Delta V_0 = \sqrt{\frac{\hbar\omega_r}{2C}}$$

Comparison to 3D cavity: go back to electric field:

$$E_0 = \frac{V_0}{b}$$

 b

Electrodynamics: energy stored in (vacuum) mode $\hbar\omega_r(\langle n \rangle + \frac{1}{2})$ $\langle n \rangle = 0$

$$\frac{1}{2} \frac{1}{2} \hbar\omega_r = \frac{\varepsilon_0}{2} \int \langle E^2 \rangle_0 \, d^3x \qquad \varepsilon_0 = 9 \times 10^{-12} \frac{C}{Vm}$$

magnetic field

$$\langle E^2 \rangle_0 = E_0^2 \langle 0 | \underbrace{(a + a^\dagger)^2}_{1} | 0 \rangle$$

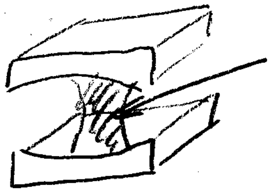
$$\frac{\varepsilon_0}{2} \int \langle \hat{E}^2 \rangle_0 \, d^3x = \left| \hat{E}(x) = \hat{E} \right| = \frac{\varepsilon_0}{2} E_0^2 V = \frac{1}{4} \hbar\omega_r$$

mode volume

$$\Rightarrow E_0 = \sqrt{\frac{\hbar\omega_r}{2\varepsilon_0 V}} \quad \alpha$$

②

V in 3D cavity:



$\sim 700\,\mathrm{mm}^3$

$\omega_1 = 2\pi\, 50\,\mathrm{GHz}$

V in 1D transmission line : $10\,\mathrm{mm} \times 10^{-2}\,\mathrm{mm} \times 10^{-4}\,\mathrm{mm} = 10^{-5}\,\mathrm{mm}^3$

ω_1 —— " —— $\sim 2\pi\, 10\,\mathrm{GHz}$

$$\sqrt{\frac{50}{700}} \sim 0.3 \quad \text{vs.} \quad \sqrt{\frac{10}{10^{-5}}} \sim 10^3$$

$\Rightarrow$ $\boxed{\text{Factor } 10^3 - 10^4 \text{ larger}}$

Jaynes Cummings for circuit QED:

1) Two-level approximation of CPB Hamiltonian

$$H = \sum_N \left\{ E_C (N - N_g)^2 |N\rangle\langle N| - \frac{E_J}{2} \left(|N\rangle\langle N+1| + |N+1\rangle\langle N| \right) \right\}$$

take only $N = 0, 1$ into account:

$$H_2 = E_C N_g^2 |0\rangle\langle 0| + E_C (1 - N_g)^2 |1\rangle\langle 1| - \frac{E_J}{2} \underbrace{\left(|0\rangle\langle 1| + |1\rangle\langle 0| \right)}_{\bar\sigma_x}$$

$$= -\frac{E_{el}}{2} |0\rangle\langle 0| + \frac{E_{el}}{2} |1\rangle\langle 1| - \frac{E_J}{2} \bar\sigma_x$$

$\Rightarrow$ shift of energy: $E_{el} = E_C (1 - 2N_g)$

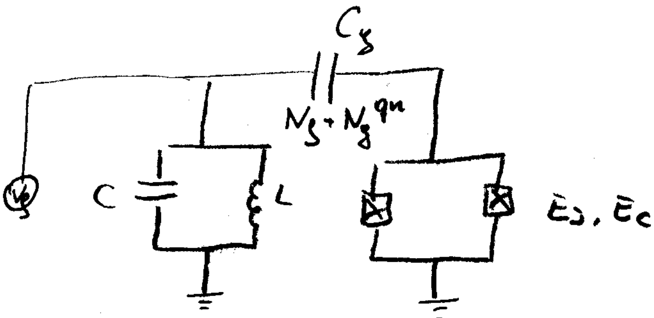
$$\Rightarrow = -\frac{E_{el}}{2} \bar\sigma_z - \frac{E_J}{2} \bar\sigma_x$$

$\Rightarrow$ Eigenbasis by rotation about y-axis: $e^{i\frac{\theta}{2}\sigma_y}$: $\sigma_x = \cos\theta\,\bar\sigma_x + \sin\theta\,\bar\sigma_z$, $\sigma_z = -\sin\theta\,\bar\sigma_x + \cos\theta\,\bar\sigma_z$

$$H_2 = \frac{1}{2} \begin{pmatrix} -E_{el} & -E_J \\ -E_J & E_{el} \end{pmatrix}$$

$\Rightarrow H_2 = \frac{\hbar\Omega}{2} \sigma_z$ $\qquad \theta = \arctan\left(\frac{E_J}{E_{el}}\right)$

2) Coupling to gate capacitor:



$$H = \frac{1}{2} E_C \left(1 - 2(N_g + \hat N_g^{qu})\right) \bar\sigma_z - \frac{E_J}{2} \bar\sigma_x$$

For simplicity $N_g = \frac{1}{2}$: $\quad H = \frac{E_C}{2} \left(\hat N_g^{qu} \bar\sigma_z - \frac{E_J}{2} \bar\sigma_x \right) =$

$$\hat Q^{qu} = (2e) \hat N_g^{qu} = C_g \hat V_g \qquad = \frac{E_C}{2} \underbrace{\frac{C_g}{2e} \sqrt{\frac{\hbar\omega}{2C}} \left(\hat a^\dagger + \hat a \right)}_{C_g \hat V_g / 2e} \bar\sigma_z - \frac{E_J}{2} \bar\sigma_x$$

$\theta = \arctan\left(\frac{E_J}{E_L}\right) \qquad E_L \sim 0 \; @ N_g = \frac{1}{2} \quad \theta = \frac{\pi}{2}$ ⑤

$$E_C = \frac{(2e)^2}{2C_\Sigma}$$

$\Rightarrow \quad \sigma_x = \bar{\sigma}_z \qquad \sigma_z = -\bar{\sigma}_x$

$$= e\,\frac{C_g}{C_\Sigma}\sqrt{\frac{\hbar\omega_1}{2C}}\,(a^+ + a)\,\underbrace{\sigma_x}_{\substack{\parallel \\ (\sigma^+ + \sigma^-)}} + \frac{E_J}{2}\sigma_z$$

$$\sigma^+ = \sigma_x + i\sigma_y$$
$$\sigma^- = \sigma_x - i\sigma_y$$

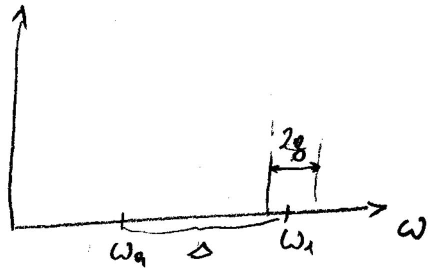
$\cancel{a^+\sigma^+} + a\sigma^+ + a^+\sigma^- + \cancel{a\sigma^-}$ energy conservation, RWA

$\Rightarrow$ full circuit Hamiltonian

$$\boxed{\hat{H} = \hbar\omega_1\left(a^+a + \frac{1}{2}\right) + \frac{E_J}{2}\sigma_z + \underbrace{\frac{C_g}{C_\Sigma}2e\sqrt{\frac{\hbar\omega_1}{2C}}}_{\hbar g}\left(a\sigma^+ + a^+\sigma^-\right)}$$

$\frac{2g}{\hbar}$ = Vacuum Rabi Frequency

Dispersive limit: $|\Delta| = |\omega_a - \omega_r| \gg g \Rightarrow \frac{g}{\Delta} \ll 1$



Transform Hamiltonian:

$$U = \exp\left[\frac{g}{\Delta}(a\sigma^+ - a^\dagger\sigma^-)\right] \quad \ldots$$

clever choice to decouple qubit and resonator

$$H^D = UHU^\dagger \approx \hbar\left(\omega_r + \frac{g^2}{\Delta}\sigma_z\right)a^\dagger a + \frac{\hbar}{2}\left(\omega_a + \frac{g^2}{\Delta}\right)\sigma_z$$

$H = H_0 + \lambda H_1 \qquad \lambda$ small

Transform $H \rightarrow \tilde{H}$ with $e^{\lambda S} H e^{-\lambda S}$ such that

$$[H_0, S] = H_1$$

$$e^{\lambda S} H e^{-\lambda S} = H + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \underbrace{[S, H]_{(n)}}_{[S,[S,[\ldots[S,H]]]]} =$$

$$[S,[S,[\ldots[S,H]]]]$$ (n-times)

$$= H_0 + \lambda H_1 + \underbrace{\lambda[S, H]}_{-\underbrace{\lambda[S,H_0]}_{-\lambda H_1} + \lambda^2[S,H_1] + \mathcal{O}(\lambda^2)} + \mathcal{O}(\lambda^2)\ldots$$

$$= H_0 + \mathcal{O}(\lambda^2)$$