2010

2a)

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle \sigma_x \rangle = \langle \psi | \sigma_x | \psi \rangle = \begin{cases} \langle 1 | \sigma_x | 1 \rangle = \underline{\underline{0}} \\ \frac{1}{2}\left(\langle 0| + \langle 1|\right) \sigma_x \left(|0\rangle + |1\rangle\right) = \\ \frac{1}{2}\Big(\underbrace{\langle 0 | \sigma_x | 0 \rangle}_{0} + \underbrace{\langle 1 | \sigma_x | 0 \rangle}_{1} + \\ \quad \underbrace{\langle 0 | \sigma_x | 1 \rangle}_{1} + \underbrace{\langle 1 | \sigma_x | 1 \rangle}_{0} = \underline{\underline{1}} \end{cases}$$

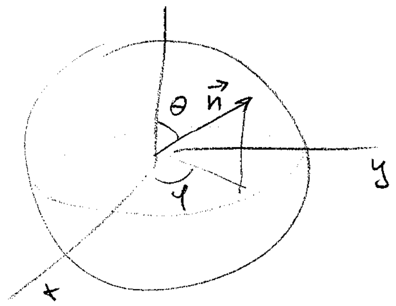
$$\langle \sigma_z \rangle = \langle \psi | \sigma_z | \psi \rangle = \begin{cases} \langle 1 | \sigma_z | 1 \rangle = \langle 1 | 1 \rangle\langle 1 | 1 \rangle - \\ \qquad \langle 1 | 0 \rangle\langle 0 | 1 \rangle = \\ \qquad = 1 \\ \frac{1}{2}\left(\langle 0| + \langle 1|\right) \sigma_z \left(|0\rangle + |1\rangle\right) = \\ \frac{1}{2}\Big(\underbrace{\langle 0 | \sigma_z | 0 \rangle}_{-1} + \underbrace{\langle 1 | \sigma_z | 0 \rangle}_{0} + \underbrace{\langle 0 | \sigma_z | 1 \rangle}_{0} + \\ \quad \underbrace{\langle 1 | \sigma_z | 1 \rangle}_{1} = \underline{\underline{0}} \end{cases}$$

Uncertainty: $\Delta\sigma = \langle \sigma^2 \rangle - \langle \sigma \rangle^2$

$$\sigma_x : \begin{cases} \langle 1 | \sigma_x^2 | 1 \rangle - 0 = \underline{\underline{1}} \text{ .. max. uncertainty} \\ \langle x_+ | \sigma_x^2 | x_+ \rangle - 1 = 1 - 1 = 0 \text{ ... no uncertainty} \end{cases}$$

$$\sigma_z : \begin{cases} \langle 1 | \sigma_z^2 | 1 \rangle - 1 = \underline{\underline{0}} \\ \langle x_+ | \sigma_z^2 | x_+ \rangle - 0 = \underline{\underline{1}} \end{cases}$$

2b) $\sigma_n = \vec{n}\cdot\vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$



$$= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$$

$\Rightarrow$ eigenstate to σ_n : $P_n = \frac{1}{2}(1 + \vec{n}\cdot\vec{\sigma})$

$$\psi_n = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix}$$

$$\sigma_n |\psi_n\rangle = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2} \\ e^{i\varphi}\left(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right) \end{pmatrix} =$$

$$\cancel{= \begin{pmatrix} \cos^2\frac{\theta}{2} - \sin^2\theta/2 + 2\sin^2\theta/2\cos\theta/2 \\ e^{i\varphi}(\end{pmatrix}}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$ ✓

3a) $|\psi\rangle = a|0\rangle + b|1\rangle$

a, b complex numbers

$a = \mathrm{Re}\,a + i\,\mathrm{Im}\,a \qquad b = \mathrm{Re}\,b + i\,\mathrm{Im}\,b$

4 real Coefficients $\Rightarrow$ 4 measurements are needed

But: x) normalization is known to be 1

$$\langle\psi|\psi\rangle = (a^*\langle 0| + b^*\langle 1|)(a|0\rangle + b|1\rangle)$$

$$= |a|^2 + |b|^2 = 1$$

x) global phase is irrelevant since it cannot be measured

$$\langle\psi|\hat{P}|\psi\rangle \Rightarrow \langle\tilde{\psi}|\hat{P}|\tilde{\psi}\rangle \quad \text{with } |\tilde{\psi}\rangle = e^{i\varphi}|\psi\rangle$$

$$= \langle\psi|e^{-i\varphi}\hat{P}e^{i\varphi}|\psi\rangle$$

$$= \langle\psi|\hat{P}|\psi\rangle$$

$\Rightarrow$ only 2 measurements are needed!

3b) 1. Measurement: probability to find system in ground state $p_{z+} = \langle\psi| P_{z+} |\psi\rangle =$

$$= \langle\psi| 0\rangle\langle 0 |\psi\rangle$$

$$= (a^* \langle 0| + b^* \langle 1|) |0\rangle\langle 0| (a|0\rangle + b|1\rangle) = |a|^2$$

2. Measurement: prob. to find system in state $|x+\rangle$:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$p_{x+} = \langle\psi| \hat{P}_{x+} |\psi\rangle = \frac{1}{2}\left(a^* \langle 0| + b^* \langle 1|\right)\left(|0\rangle + |1\rangle\right) \cdot$$

$$\left(\langle 0| + \langle 1|\right)\left(a|0\rangle + b|1\rangle\right)$$

$$= \frac{1}{2}(a^* + b^*)(a+b) = \frac{1}{\sqrt{2}}\left(|a|^2 + |b|^2 + 2\mathrm{Re}\, a^* b\right)$$

$\Rightarrow$ 4 equations to determine state

1) $|a|^2 + |b|^2 = 1$ (norm)

2) $\mathrm{Im}(a) = 0$ (global phase)

3) $\frac{1}{2}\left(|a|^2 + |b|^2 + 2\mathrm{Re}\, a^* b\right) = \{1\}, \{2\}$

$$= \frac{1}{2}\left(1 + 2a\, \mathrm{Re}\, b\right)$$

4) $p_{z+} = |a|^2$

(3) + (4): $\mathrm{Re}(b) = (2p_{x+} - 1) / 2\sqrt{p_{z+}}$

$\mathrm{Im}(a) = 0$

$\mathrm{Re}(a) = \sqrt{p_{z+}}$

(1): $a^2 + (\mathrm{Re}\, b)^2 + (\mathrm{Im}\, b)^2 = 1 \Rightarrow \mathrm{Im}\, b = \sqrt{1 - p_{z+} - \frac{(2p_{x+} - 1)^2}{4p_{z+}}}$

e.g.: $|y_+\rangle = \frac{1}{2}(|0\rangle + i|1\rangle)$: $p_{x+} = \langle y_+ | P_{x+} | y_+ \rangle =$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\left(\langle 0| - i\langle 1|\right)\left(|0\rangle + |1\rangle\right)\left(\langle 0| + \langle 1|\right)\left(|0\rangle + i|1\rangle\right)$$

$$= \frac{1}{4}\cdot(1-i)(1+i) = \frac{1}{2}$$

$$p_{z+} = \langle y_+ | P_{z+} | y_+ \rangle = \frac{1}{2}\left(\langle 0| - i\langle 1|\right)|1\rangle\langle 1|\left(|0\rangle + i|1\rangle\right) = \frac{1}{2}$$

$$\Rightarrow \mathrm{Re}(a) = \frac{1}{\sqrt{2}}$$

$$\mathrm{Im}(a) = 0$$

$$\mathrm{Re}(b) = \frac{2p_{x+} - 1}{2\sqrt{p_z}} = 0$$

$$\mathrm{Im}(b) = \sqrt{1 - \frac{1}{2} - 0} = \frac{1}{\sqrt{2}}$$

3c) Normalisation: # of measurement decreases by 1 (one more condition on the state)

can only be used, if there is no other possibility to find the system in another state

eg.: if $|\psi\rangle = a|0\rangle + b|1\rangle + c|2\rangle$

$\underbrace{a|0\rangle + b}_{}$ not normalized to $1 + |a|^2 + |b|^2$

$\rightarrow$ another measurement is required.