The Deutsch-Josza Algorithm in NMR

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Introduction

- Algorithm presented in 1992 by Deutsch and Josza
- First implementation in 1998 on **NMR system**:
  - Jones, JA; Mosca M; et al.
    Implementation of a quantum algorithm on a nuclear magnetic resonance quantum computer.
    J. Chem. Phys. 109, 1648 (1998)
  - Chuang, IL; Vandersypen, LMK; Zhou, X; et al.
    Experimental realization of a quantum algorithm.
    Nature 393, 143 (1998)
- Little practical use, but **first demonstration of a quantum algorithm** that performs a task more efficiently than any classical one

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Deutsch Algorithm: Task

In General:

Given unknown function f: $\{0,1\}^N \longrightarrow \{0,1\}$

- f is *constant* if $f(x) = 0$ or $f(x) = 1 \, \forall x$
- f is *balanced* if $f(x) = 0$ for half the input and $f(x) = 1$ for other half of input

Simple case:

f: $\{0,1\} \longrightarrow \{0,1\}$ Input is 1 bit

$$\text{4 possibilities} \begin{cases} f_1(x) = 0, & \text{constant} \\ f_2(x) = 1, & \text{constant} \\ f_3(x) = x, & \text{balanced} \\ f_4(x) = \text{NOT}\, x, & \text{balanced} \end{cases}$$

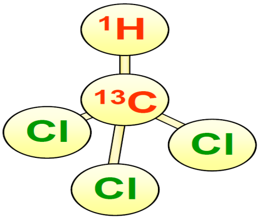
Analogous task: Determine whether a coin is fair or fake

The System Hamiltonian

$$\mathcal{H}_{NMR} = \mathcal{H}_{sys} + \mathcal{H}_{control} + \mathcal{H}_{env}$$

where

$$\mathcal{H}_{sys} = -\omega_A I_A^z - \omega_B I_B^z + 2\pi J\, I_A^z\, I_B^z$$



- free precession about $-\vec{B}_0$
- $\omega_A \simeq 500\,\mathrm{MHz}$, $\omega_B \simeq 125\,\mathrm{MHz}$ $\rightarrow$ single qubit addressing (heteronuclear spins: $^{1}\mathrm{H}$ (A) and $^{13}\mathrm{C}$ (B) ; rf range)
- scalar spin-spin coupling $\rightarrow$ two-qubit gates

The Control and the Environment Hamiltonians

$$\mathcal{H}_{control} = -\sum_{i=A,B} \hbar\omega_i \left[\cos(\omega_{rf} t + \phi) I_i^x - \sin(\omega_{rf} t + \phi) I_i^y\right]$$

where $\omega_i = \gamma_i B_{rf}$ $\rightarrow$ shifting to the rotating frame, $\mathcal{H}_{rot}$ is no longer t dependent

$\mathcal{H}_{env}$ leads to decoherence (good in NMR):

- spin-spin couplings, fluctuating $\vec{B}$ fields $\rightarrow$ $T_2 \simeq 1\,\mathrm{s}$ $\rightarrow$ "elastic scattering"
- couplings spins-lattice $\rightarrow$ $T_1 \simeq 10^2 \div 10^3\,\mathrm{s}$ $\rightarrow$ "inelastic scattering"

Single qubit and two-qubit gates

- quantum control $\rightarrow$ $U = U_k U_{k-1} \dots U_2 U_1$
- $\forall$ single-qubit gate $U \rightarrow U = e^{i\alpha} R_x(\beta) R_y(\gamma) R_x(\delta)$

  where $R_{\hat{n}}(\theta) = \exp\left[-i \frac{\theta\, \hat{n}\cdot\vec{\sigma}}{2}\right]$
- rf fields $\rightarrow$ $U_{rf} = \exp\left[i\omega(\cos\phi\, I^x - \sin\phi\, I^y) t_{pw}\right]$
  where t_{pw} is the *pulse width*

  for example, $R_z(\theta) = X\, R_y(\theta)\, \bar{X}$
- pulse sequence design
- two-qubit gates $\rightarrow$ J coupling :

$$U_{CNOT} = \sqrt{i}\, Z_A\, \bar{Z}_B\, X_B\, U_J(1/2J)\, Y_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Experimental limitations

cross-talk

- pulse on resonance with qubit A perturbs the state of qubit B (frequency bandwidth $\propto 1/t_{pw} \rightarrow$ wide frequency range excitation)
- solution: from "hard" (rectangular) pulses, to "soft" pulses $\rightarrow$ increase t_{pw}, but: decoherence effects get worse

off-resonance effects

- J coupling always present $\rightarrow$ frequency shifting to $\omega_i \pm \pi J$ turning the coupling off when undesired: "refocusing" $\rightarrow X_A^2\, U_J(\tau)\, X_A^2 = U_J(-\tau)$

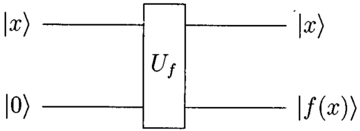
Deutsch Algorithm: Quantum-mechanical Approach

Formally: Evaluate $f(0) \oplus f(1)$ (addition modulo 2)

- Classically: Try twice with both inputs
- QM: Try once with a superposition as input

2-qubit system with 1 **input-qubit** and 1 **work-qubit**
To have reversible process: Unitary operation U_f that contains f

$$|x\rangle|y\rangle \xrightarrow{U_f} |x\rangle|y \oplus f(x)\rangle$$

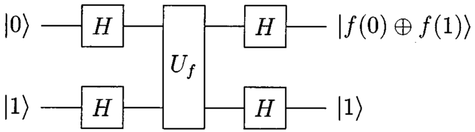


Way how to evaluate f(x) classically on a quantum computer

Deutsch Algorithm: Quantum-mechanical Approach

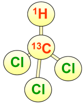
Use **superposition as input**, achieved with Hadamard gates

$$\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \xrightarrow{U_f} (-1)^{f(0)}\left(\frac{|0\rangle + (-1)^{f(0)\oplus f(1)}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

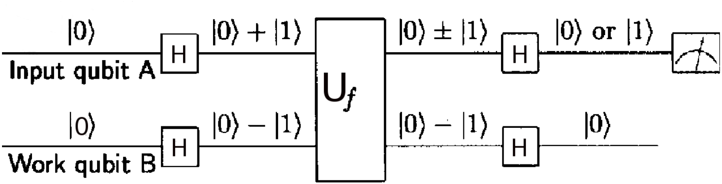


- Value of $f(0) \oplus f(1)$ is stored in relative phase of first qubit
- Convert superposition back into eigenstates at the end with Hadamard gates

Deutsch Algorithm on NMR: Initialisation



- $\mathrm{CHCl_3}$ molecules in acetone solvent
- At room temperature, therefore use 'temporal averaging' (3 measurements after permutations)



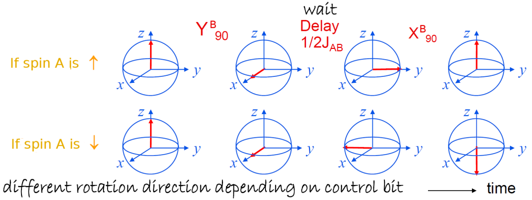
- $\pi/2$ rotations in Bloch sphere with pulses in x-y plane
- Radiofrequency pulse in y-direction rotates vector about y-axis

→ Way to implement Hadamard

Deutsch Algorithm on NMR: Implementing U_f

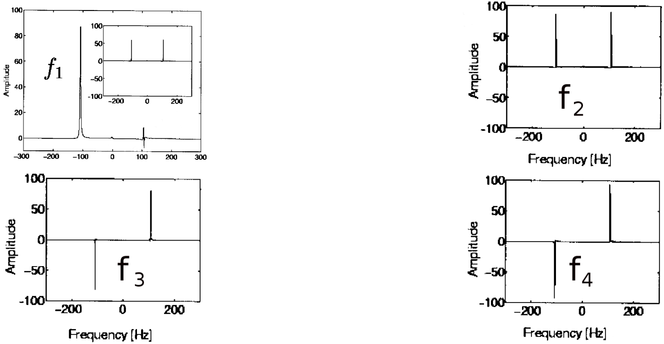
x is input qubit A

- $f_1(x) = \mathbf{0}$: Do nothing (identity)
- $f_2(x) = \mathbf{1}$: Do always spin flip (180° pulse) on qubit B
- $f_3(x) = x$: Flip qubit B when qubit A is in state $|1\rangle$ (CNOT)
- $f_4(x) = \mathbf{NOT}\, x$: Flip qubit B when qubit A is in state $|0\rangle$



Attention: In the experiment CNOT operates on superpositions

Deutsch Algorithm on NMR: Read-out



- Left frequency corresponds to qubit A, right one to B
- $|0\rangle$ gives emission (>0), $|1\rangle$ absorption (<0)
- Work qubit always unchanged, qubit A is $|0\rangle$ when const. and $|1\rangle$ when balanced

integer N=15 factorization with the Shor algorithm

- l-digit integer: from $O(2^{l^{1/3}})$ to $O(l^3)$ operations
- turning the factoring problem into a period finding problem $f(x) = a^x \mod N$
- find the period by QFT
- obtaining factors from period

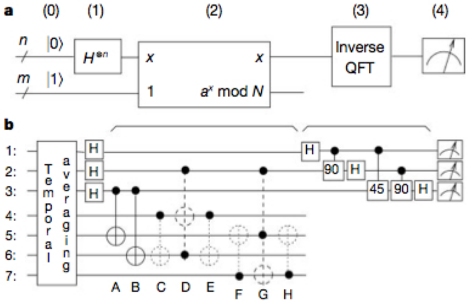


Figure: **a**: outline of the quantum circuit. **b**: detailed N=15, a=7 case

Conclusion

- Pulse techniques to implement quantum control
- NMR showed promise as a technology for the implementation of small QC (D-J alg.)
- Positive and negative aspects of NMR (q. coherence/exp. limitations, giant molecules?)
- Greatest result in QC through NMR $\rightarrow$ factoring

List of the references used

- L. M. K. Vandersypen, I.L. Chuang *NMR techniques for quantum control and computation*, Reviews of Modern Physics, 76 (2004).
- L. M. K. Vandersypen et. Al. *Experimental realization of Shor's quantum factoring algorithm using nuclear magnetic resonance*, Nature, 414 (2001).
- I.L. Chuang, L. M. K. Vandersypen, X. Zhou, D.W. Leung, S. Lloyd, *Experimental realization of a quantum algorithm*, Nature, 393 (1998).
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