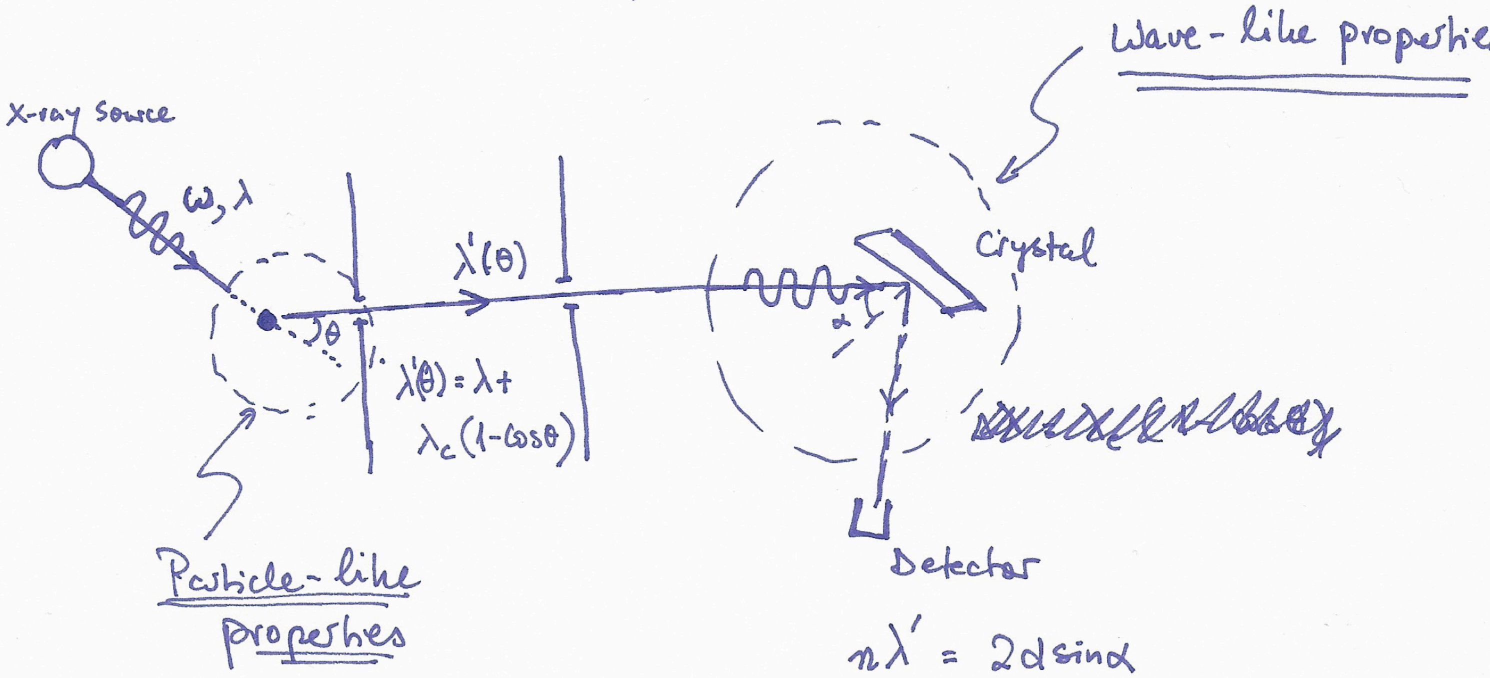
Photons → display wave-like and particle-like properties

best example: Compton experiment



$$n\lambda' = 2d\sin\alpha$$

$$\left.\begin{aligned} E &= \hbar\omega \\ p &= \frac{E}{c} = \frac{\hbar\omega}{c} = \hbar k \quad \left(= \frac{h}{\lambda}\right) \end{aligned}\right\} \Rightarrow \boxed{\omega = ck}$$

"Dispersion relation"

Consequence of energy and momentum being linearly related $E = c \cdot p$

Particles de Broglie: Same relations for (free) particles?

$$\left.\begin{aligned} E &= \hbar\omega \\ p &= \frac{h}{\lambda} \end{aligned}\right\}$$

But here $E = \frac{p^2}{2m}$ so that

$$\omega(k) = \frac{\hbar}{2m}k^2$$

Discuss the different dispersion relation wrt EM waves.

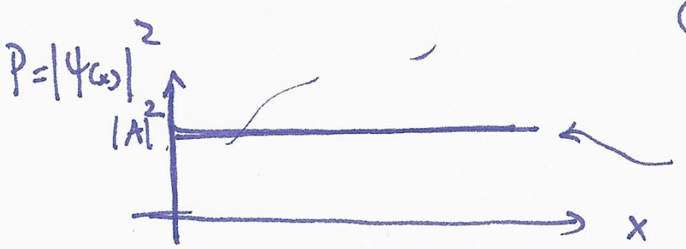
Proof. ⇒ Davisson – Germer Expt. (short discussion)

Simplest case: Free particle with a definite momentum p and energy $E = p^2/2m$.

$$\psi(x,t) = \underset{\substack{\Downarrow \\ z = re^{i\phi} = x+iy}}{A\, e^{i(kx-\omega t)}} \rightarrow \text{complex!}$$

Born: $|\psi(x,t)|^2$:→ associate to the probability of particle to be found at (x,t)

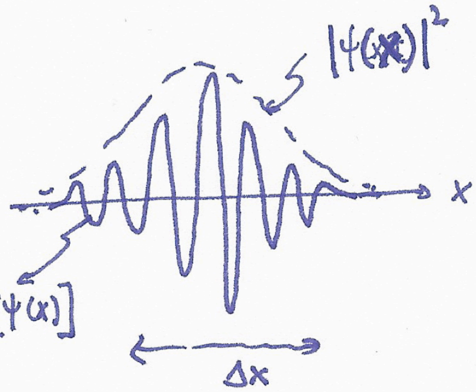
t=0 :



Particle position is completely indeterminate?!

Consider at $t=0$ a situation with particle position specified ~~to~~ within Δx. Example:

$$\psi(x) = e^{-x^2/2\Delta x^2}\, e^{ik_0 x}$$



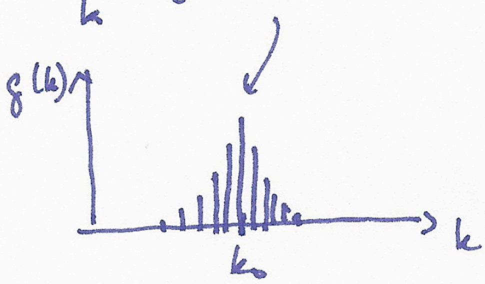
But now λ is not well-defined!
(visual)

What is the momentum content?

$$\psi(x) = \overbrace{(1/\pi\Delta x^2)^{1/4}}^{\rightarrow \text{normalization}}\, e^{-x^2/2\Delta x^2}\, e^{ik_0 x} = \sum_k \underbrace{g(k)}_{\text{weights}}\, e^{-ikx}$$

Fourier transform:

Diagnostic tool for momentum content



$\psi(x) = \int_{-\infty} dk \; e \quad g(k)$: Synthesis

Computation $\rightarrow$ $g(k) = \int_{-\infty}^{\infty} dk \; e^{ikx} f(x)$

$$= e^{-\Delta x^2 \cdot (k-k_0)^2/2} \cdot \left(\frac{\Delta x^2}{\pi}\right)^{1/4}$$

$$\Leftrightarrow e^{-(k-k_0)^2/2(\Delta k)^2}$$

$$\Rightarrow \underbrace{\Delta k}_{\equiv \text{ momentum spread}} \underbrace{\Delta x}_{\equiv \text{ position spread}} = 1 \quad !$$

≡ momentum spread ($p = \hbar k$)

≡ position spread

$$\Rightarrow \boxed{\Delta p \, \Delta x = \hbar}$$

This is specific for a gaussian envelope, for any other shape

$$\Delta p \, \Delta x \geq \hbar$$

"Heisenberg uncertainty Relation" $\rightarrow$ Discuss but more later.

$\therefore$ Uncertainty relation is a direct consequence of fourier analysis. ($e^{ikx} \Leftrightarrow$ momentum operator $\Leftrightarrow$ Fourier transform $\int dk \, e^{ikx}$)

This is all at $t=0$. Now let time flow...

Important link: dispersion relation

$$k \to \omega(k)$$

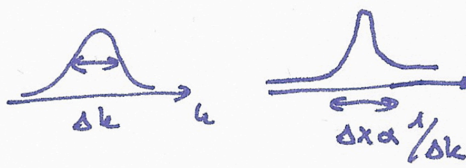
$$\psi(x,0) = \int_{-\infty}^{\infty} dk\, g(k)\, e^{ikx}$$

$$\psi(x,t) = \int_{-\infty}^{\infty} dk\, g(k)\, e^{i(kx - \omega(k)t)}$$

Here Schrödinger particle or EM wave matters

$$\omega \propto k \quad , \quad \omega \propto k^2$$

Let $g(k) = e^{-(k-k_0)^2/2\Delta k^2}$



$$k = k_0 + \delta k$$

$$\omega(k) = \omega(k_0) + \overbrace{\frac{d\omega}{dk}\Big|_{k_0}}^{v_g} \delta k + \frac{1}{2} \overbrace{\frac{d^2\omega}{dk^2}\Big|_{k_0}}^{\sigma} \delta k^2$$

$$\psi(x,t) \propto e^{i(k_0 x - \omega_0 t)}\, e^{-(x - v_g t)^2/2\Delta_x^2}$$

$$\Delta_x = \Delta_x(t) = \frac{1 + i\sigma(\Delta k^2)t}{\Delta k^2}$$