Physik IV - Lösungen - Serie 8

21. April 2011

1) $\hat{B}$ is measured and b_1 obtained as measurement result
→ state is 'projected' ('collapse of the wave function')
onto corresponding eigenvector / eigenfunction

$$\phi_1 = (\psi_1 + 2\psi_2)/\sqrt{5}$$

ϕ_1 is not an eigenfunction of $\hat{A}$, thus the expectation value of $\langle A \rangle$ is not sharp. Instead a measurement of $\hat{A}$ results in either a value a_1 or a_2.
The probability of either event can be calculated by calculating the overlap between the wave function and the eigenvector belonging to a_1 or a_2, respectively:

$$p(a_1) = \left| \int \psi_1^*(x)\, \phi_1(x)\, dx \right|^2 =$$

$$= \left| \frac{1}{\sqrt{5}} \int \psi_1^*(x) \left(\psi_1(x) + 2\psi_2(x) \right) dx \right|^2 =$$

$$= \frac{1}{5} \left| \underbrace{\int \psi_1^*(x)\, \psi_1(x)\, dx}_{1} + \frac{2}{5} \underbrace{\int \psi_1^*(x)\, \psi_2(x)\, dx}_{0} \right|^2 =$$

orthogonality:

$$= \underline{\underline{\frac{1}{5}}}$$

$$p(a_2) = \left| \int \psi_2^*(x)\, \phi_1(x)\, dx \right|^2 = \ldots = \underline{\underline{\frac{4}{5}}}$$

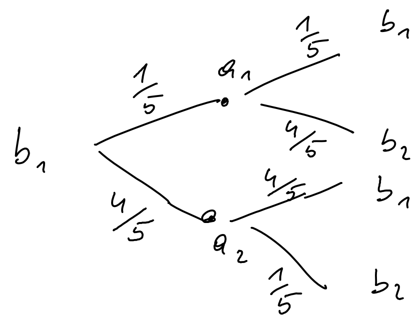
after the measurement the system is in the state
ψ_1 (with probability $\frac{1}{5}$) or ψ_2 (with probability $\frac{4}{5}$)

A subsequent measurement of $\hat{B}$ projects the system onto an eigenstate of $\hat{B}$, i.e. ϕ_1 or ϕ_2.

With
$$\psi_1 = (\phi_1 + 2\phi_2)/\sqrt{5} \quad \text{and}$$
$$\psi_2 = (2\phi_1 - \phi_2)/\sqrt{5}$$

we can again calculate the conditional probabilities

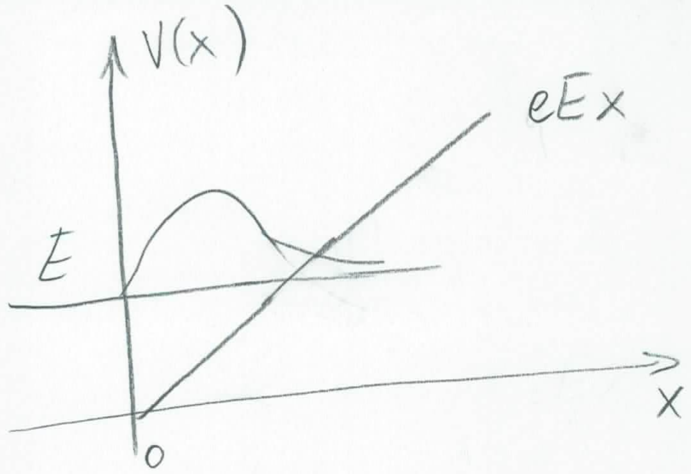
$$p(b_1|a_1) = \left| \int \phi_1^*(x)\, \psi_1(x)\, dx \right|^2 = \frac{1}{5}$$
$$p(b_1|a_2) = \left| \int \phi_1^*(x)\, \psi_2(x)\, dx \right|^2 = \frac{4}{5}$$
$$p(b_2|a_1) = \left| \int \phi_2^*(x)\, \psi_1(x)\, dx \right|^2 = \frac{4}{5}$$
$$p(b_2|a_2) = \left| \int \phi_2^*(x)\, \psi_2(x)\, dx \right|^2 = \frac{1}{5}$$



In total, the probability to obtain b_1 after the measurement sequence $\hat{B} \cdot \hat{A} \cdot \hat{B} = \frac{1}{5} \cdot \frac{1}{5} + \frac{4}{5} \cdot \frac{4}{5} = \underline{\underline{\frac{17}{25}}}$

Problem 2

a)



Schrödinger equation:

$$(*) \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + qEx\,\psi = E_n \psi \quad ; \text{ for } x>0$$

boundary conditions: $\psi(x=0)=0 \quad (**)$

Solution of $(*)$ is Airy function:

$$\psi(x) = A\,Ai\left[\left(\frac{2m}{\hbar^2 e^2 E^2}\right)^{1/3}(eEx - E_n)\right]$$

from boundary conditions $(**)$ one has

$$\left(\frac{2m}{\hbar^2 e^2 E^2}\right)^{1/3}(eEx - E_n) = x_n \quad , n = 1, \dots N \dots$$

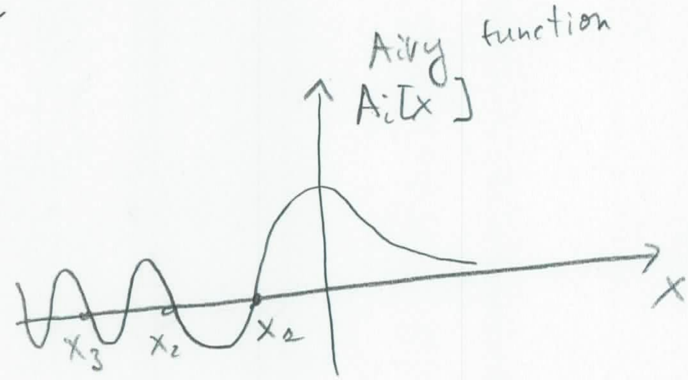
are zeros of Airy function $A_i[x]=0$.

$$\Rightarrow \quad E_n = -\left(\frac{e^2E^2\hbar^2}{2m}\right)^{1/3} x_n$$

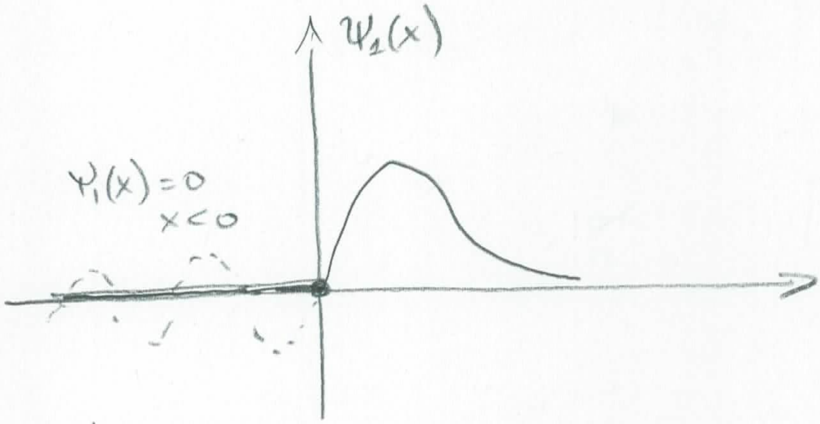
$x_1 = -2.33$

$x_2 = -4.08$

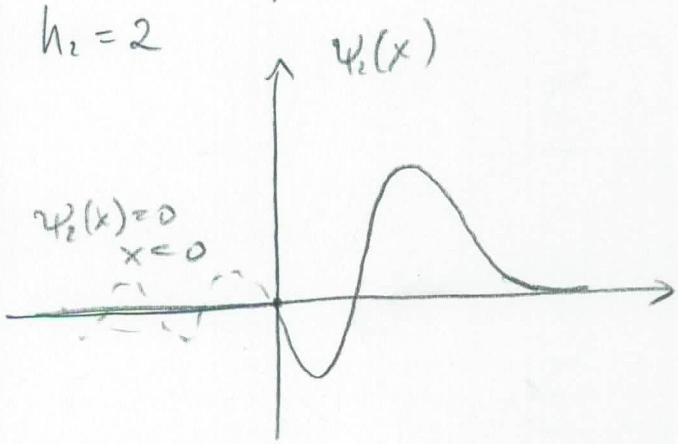
$x_3 = \dots$



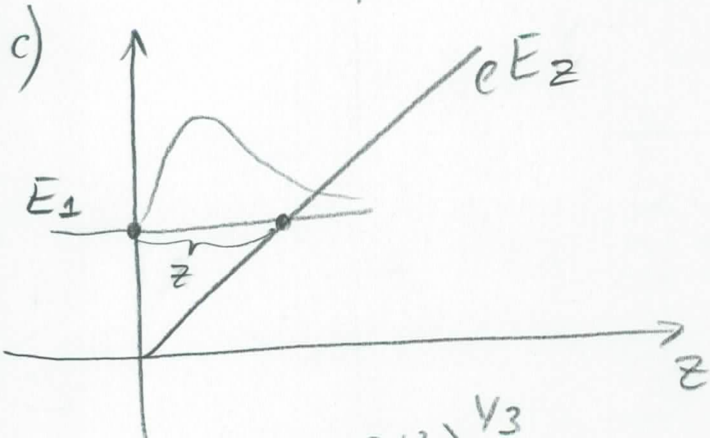
b) $n=1$ $\quad \psi(x) \propto Ai\left[\left(\frac{2m}{\hbar^2 e^2 E^2}\right)^{1/3}\left[eEx - x_1\right]\right]$



$n_2 = 2$



c)



take point $\tilde{z}_1$: $eE\tilde{z}_1 = E_1$ as characteristic confinement of $\bar{e}$ in z direction

$$\tilde{z}_1 = \frac{E_1}{eE} \Rightarrow$$

$$\tilde{z}_1 = \left(\frac{e^2 E^2 \hbar^2}{2m}\right)^{1/3} 2.33 \Big/ e \cdot 10^7\ V/m \simeq 3.65\ nm$$

$$E_2 - E_1 = \left(\frac{e^2 E^2 \hbar^2}{2m}\right)^{1/3}(4.08 - 2.33) = 4.37 \cdot 10^{-21}\ J$$

Thermal energy at room temperature

$E_{th} = k_B T = 1.38 \cdot 10^{-23} \cdot 300k = 4.14 \cdot 10^{-21} J$

$E_2 - E_1$ is comparable to E_{th} so other levels will be aslo occupied → not true 2DEG

at lower temperature $E_2 - E_1 \gg k_B T \Rightarrow$ will be true 2DEG

3) Da nur ein Energiequant $\hbar\omega$ zur Verfügung steht, kann der Zustand als Superposition von Grund- & 1. angeregten Zustand geschrieben werden:

$$\psi(x) = a\, u_0(x) + b\, u_1(x)$$

Da $\psi(x)$ normiert sein muss, d.h. $\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx = 1$ ergibt sich die folgende Bedingung für die Koeffizienten:

$$\int_{-\infty}^{\infty} \left(a^* u_0^*(x) + b^* u_1(x)\right)\left(a\, u_0(x) + b\, u_1(x)\right) dx =$$

$$= \int_{-\infty}^{\infty} |a|^2 |u_0(x)|^2 dx + \int_{-\infty}^{\infty} |b|^2 |u_1(x)|^2 dx + \left[\int_{-\infty}^{\infty} a^* b\, u_0^*(x)\, u_1(x) + c.c.\right]$$

$$= |a|^2 + |b|^2 = 1 \qquad \text{da } \int_{-\infty}^{\infty} u_j^*(x)\, u_k(x)\, dx = \delta_{jk}$$

Damit können wir ansetzen: $a = \cos\theta\, e^{i\alpha}$ $\quad b = \sin\theta\, e^{i\beta}$
Die Aufenthaltswahrscheinlichkeit im Bereich $x>0$ ist durch

$$W = \int_0^{\infty} |\psi(x)|^2 dx \quad \text{gegeben.}$$

Daraus läßt sich schließen, dass eine „globale" Phase irrelevant ist, da $|\psi(x)|^2 = |\psi'(x)|^2$ mit $\psi'(x) = e^{i\phi}\psi(x)$ $\phi \in \mathbb{R}$.

Wir können also $\psi'(x) = e^{-i\alpha}\psi(x)$ verwenden und erhalten mit $\gamma = \beta - \alpha$

$$\psi'(x) = \cos\theta\, u_0(x) + e^{i\gamma}\sin\theta\, u_1(x).$$

Es folgt:

$$W = \int_0^{\infty} |\psi'(x)|^2 dx = \cos^2\theta \int_0^{\infty} |u_0(x)|^2 dx + \sin^2\theta \int_0^{\infty} |u_1(x)|\, dx +$$

$$+ \cos\theta \sin\theta \left[e^{i\gamma} \int_0^{\infty} u_0^*(x)\, u_1^*(x)\, dx + c.c. \right]$$

mit $u_0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega_0}{2\hbar}x^2}$ und $u_1(x) = \sqrt{2}\left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega_0}{2\hbar}x^2} x \sqrt{\frac{m\omega_0}{\hbar}}$

$$\Rightarrow \int_0^\infty |u_0(x)|^2 dx = \int_0^\infty |u_1(x)|^2 dx = \frac{1}{2} \quad \text{und} \quad \int_0^\infty u_0^*(x)\, u_1(x)\, dx = \frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow W = \frac{1}{2}(\cos^2\theta + \sin^2\theta) + \frac{1}{\sqrt{2\pi}} \cos\theta \sin\theta \left(e^{i\gamma} + e^{-i\gamma}\right) =$$

$$= \frac{1}{2} + \sqrt{\frac{2}{\pi}} \cos\theta \sin\theta \cos\gamma$$

Maximum von W: $\frac{\partial W}{\partial \theta} = 0 : \sqrt{\frac{2}{\pi}}\left(-\sin^2\theta + \cos^2\theta\right) = 0 \Rightarrow \theta = \frac{\pi}{4} + \pi/2 \cdot n$

$n = 0, \pm 1, \dots$

$\frac{\partial W}{\partial \gamma} = 0 : -\sqrt{\frac{2}{\pi}} \cos\theta \sin\theta \sin\gamma \rightarrow \gamma = 0, \pi, 2\pi \dots$

$$\rightarrow \psi'(x) = \frac{1}{\sqrt{2}}\left(u_0(x) + u_1(x)\right)$$

To get maximum for W choose θ and γ such that $\cos\theta \sin\theta$ and $\cos\gamma$ have the same sign. $\Rightarrow$ have two solutions:

I

$\theta = \pi/4 + \pi \cdot n$

$\gamma = 2\pi m$

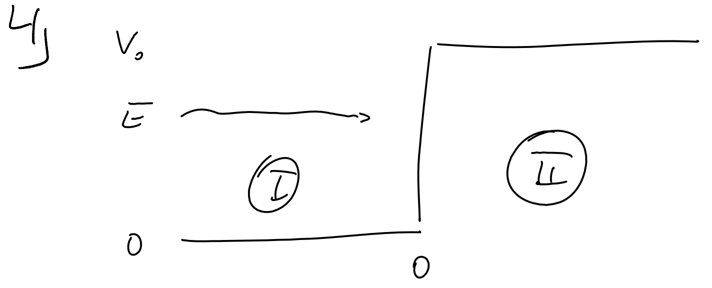
II

$\theta = \pi/4 + (2n+1)\pi/2$

$\gamma = (2m+1)\pi$

n, m are integers

4)



$H\psi(x) = E\psi(x)$ has to be fulfilled both in (I) and (II)

$$H = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

I: $$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_I(x) = E\,\psi_I(x) \qquad k_I^2 = \frac{2mE}{\hbar^2}$$

$$\frac{\partial^2}{\partial x^2}\psi_I(x) + k_I^2\,\psi_I(x) = 0$$

II: $$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_{II}(x) + V_0\,\psi_{II}(x) = E\,\psi_{II}(x)$$

$$k_{II}^2 = \frac{2m}{\hbar^2}(E - V_0)$$

$$\frac{\partial^2}{\partial x^2}\psi_{II}(x) + k_{II}^2\,\psi_{II}(x) = 0$$

<u>Ansatz</u> for the wave functions:

I: $\underbrace{e^{ik_I x}}_{\psi_I} + \underbrace{R e^{-ik_I x}}_{\psi_R}$ (incoming + reflected wave; amplitude of incoming wave is set to one, since normalization not relevant here)

II: $\underbrace{T e^{ik_{II} x}}_{\psi_T}$ (transmitted wave)

<u>Boundary conditions</u>

$\psi_I(0) = \psi_{II}(0)$: $1 + R = T$

$\psi_I'(0) = \psi_{II}'(0)$: $k_I(1 - R) = k_{II}T$

$$\Rightarrow R = 1 - \frac{k_{II}T}{k_I} = 1 - \frac{k_{II}(1+R)}{k_I}$$

$$R = \frac{1 - \frac{k_{II}}{k_I}}{1 + \frac{k_{II}}{k_I}}$$

reflection probability = $\dfrac{\text{reflected flux}}{\text{incident flux}} = \dfrac{|\psi_R|^2 \cdot v_R}{|\psi_I|^2 \cdot v_I}$

$= \dfrac{|\psi_R|^2}{|\psi_I|^2}$ since $v_R = v_I$ in region I

$= |\psi_R|^2$ since the density $|\psi_I|^2$ is unity

$$\Rightarrow |\psi_R|^2 = |R|^2 = \left| \frac{1 - k_{II}/k_I}{1 + k_{II}/k_I} \right|^2$$

for $E < V_0$: k_{II} purely imaginary while k_I is real

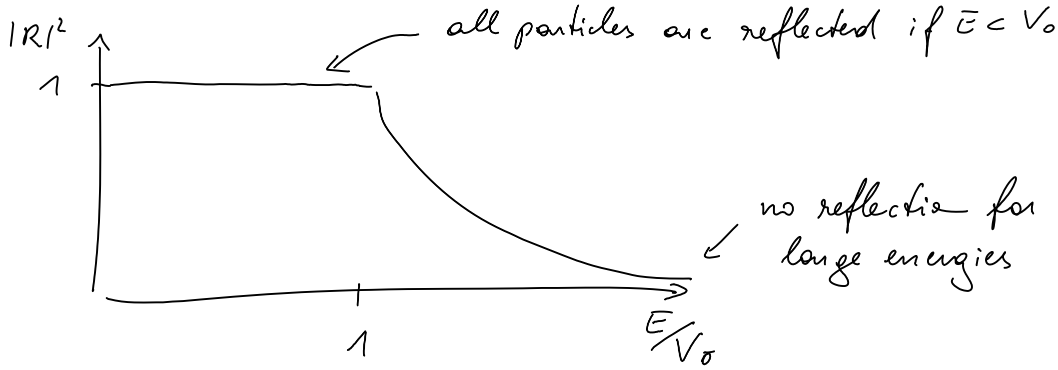
$$\Rightarrow |R|^2 = \left| \frac{a - ib}{a + ib} \right|^2 = 1$$

for $E > V_0$: both k_{II} & k_I are real

$$|R|^2 = \left(\frac{1 - k_{II}/k_I}{1 + k_{II}/k_I} \right)^2 \qquad \frac{k_{II}}{k_I} = \sqrt{\frac{E - V_0}{E}} = \sqrt{1 - \frac{V_0}{E}}$$

$$\sim 1 - \frac{V_0}{2E}$$

$$|R|^2 = \left(\frac{V_0/2E}{2 - \frac{V_0}{2E}} \right)^2 \propto \frac{1}{E} \text{ for large } E$$



b) $$|R|^2 = \left|\frac{1-\frac{k_{II}}{k_I}}{1+\frac{k_{II}}{k_I}}\right|^2 \sim \left(\frac{\frac{V_0}{2E}}{2-\frac{V_0}{2E}}\right)^2 = \left(\frac{\frac{10}{2000}}{2-\frac{10}{2000}}\right)^2 \approx \underline{6\cdot 10^{-6}}$$

$$\langle v\rangle = \frac{\langle \hat{p}\rangle}{m} = Re\int \psi^*(x)\,(-i\hbar)\frac{\partial}{\partial x}\psi(x)\,dx$$

before barrier: $\psi(x) \sim \psi_1(x)$ (neglect reflected part)

$$\langle v\rangle = \frac{(-i\hbar)\,i k_I}{m} = \frac{\hbar k_I}{m} = \frac{\hbar\sqrt{2mE}}{\hbar m} = \sqrt{\frac{2E}{m}} = 4.37\cdot 10^5\,\frac{m}{s}$$

after barrier: $\psi(x) = \psi_T(x)$

$$\langle v\rangle = \frac{\hbar k_{II}}{m} = \sqrt{\frac{2(E-V_0)}{m}} = 4.35\cdot 10^5\,\frac{m}{s}$$

classical mode: only change in velocity expected, but <u>all</u> particles are transmitted <u>without</u> any reflection.

c) penetration depth can be obtained by plugging in the (imaginary) k_{II} into the ansatz in a) for region (II)

$$\psi_T = T\cdot e^{ik_{II}x} \quad \text{with} \quad k_{II} = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} = i\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$\psi_T \propto e^{-\frac{1}{\hbar}\sqrt{2m(V_0-E)}\,x}$$

typical length scale $$x_p \sim \frac{\hbar}{\sqrt{2m(V_0-E)}} \sim \frac{\hbar}{\sqrt{2m(10-1)}} = \frac{\hbar}{\sqrt{18\cdot m\cdot e}} \sim \frac{1.6\cdot 10^{-34}}{\sqrt{18\cdot 1.6\cdot 10^{-27}\cdot 1.6\cdot 10^{-13}}} \sim 1.5\text{ fm}$$

Aufgabe 5 - SGL des Harmonischen Oszillators

```
ℏ = 1;
m = 1;
ω = 1; (* Frequenz  *)
ndivs = 300; (* Diskretisierung des Ortes*)
boundary = 10 ω; (* wieviel Platz wird ausserhalb  miteingeschlossen *)
dx = 2 boundary / ndivs; (* Schrittweite *)
D2op[A_, dim_] := DiagonalMatrix[Table[-2 A, {dim}]] +
  DiagonalMatrix[Table[1 A, {dim - 1}], 1] + DiagonalMatrix[Table[1 A, {dim - 1}], -1]
D1op[A_, dim_] := DiagonalMatrix[Table[A, {dim}]] +
  DiagonalMatrix[Table[-A, {dim - 1}], -1]


creationop[m_, ω_] := Sqrt[1/(2 m ℏ ω)]
  (DiagonalMatrix[Table[m ω x, {x, -boundary, boundary, dx}]] - ℏ 1/dx D1op[1, ndivs + 1])
annihilationop[m_, ω_] := Sqrt[1/(2 m ℏ ω)]
  (DiagonalMatrix[Table[m ω x, {x, -boundary, boundary, dx}]] + ℏ 1/dx D1op[1, ndivs + 1])
```

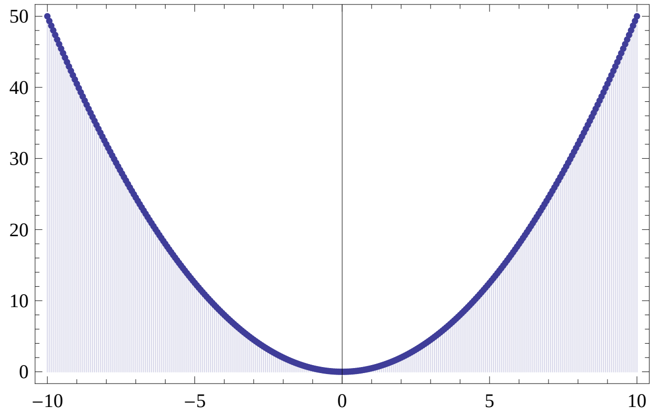
Definiere das Potential :

In[11]:=
```
VfuncHO[x_, m_, ω_] := (m ω^2 x^2)/2
```

Werteliste für Potential :

In[20]:=
```
xValues = Table[x, {x, -boundary, boundary, dx}];
```

In[21]:=
```
VarrayHO = Table[VfuncHO[x, m, ω], {x, -boundary, boundary, dx}];
ListPlot[{xValues, VarrayHO}ᵀ,
 Filling → 1.02 Min[VarrayHO], Frame → True, PlotRange → All]
```

Out[22]=


Aufstellen der Matrix:

In[14]:=
```
MProblemHO = D2op[-ℏ^2/(2 m dx^2), ndivs + 1] + DiagonalMatrix[VarrayHO] // N;
```

Lösen der Eigenwertgleichung :

In[15]:=
```
answersHO = Eigensystem[MProblemHO];
{evalsHO, evectsHO} = Sort[Transpose[answersHO], #1[[1]] < #2[[1]] &] // Transpose;
```

Berechnung des ersten angeregten Zustands aus dem Grundzustand:

In[17]:=
```
evec1 = creationop[m, ω].evectsHO[[1]];
```

Graphik und Vergleich mit ersten angeregten Zustand aus Eigenwertgleichung :

In[23]:=

```
ListPlot[{{xValues, evec1}ᵀ, {xValues, evectsHO[[2]]}ᵀ}, Joined → True,
 Filling → Axis, PlotRange → All, ImageSize → 600, Frame → True,
 BaseStyle → {FontSize → 14, FontFamily -> "Arial"}, FrameLabel → {"x", "Ψ(x)"}]
```

Out[23]=