Experimental Realization of Quantum Games on a Quantum Computer

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We generalize the quantum prisoner's dilemma to the case where the players share a nonmaximally entangled states. We show that the game exhibits an intriguing structure as a function of the amount of entanglement with two thresholds which separate a classical region, an intermediate region, and a fully quantum region. Furthermore this quantum game is experimentally realized on our nuclear magnetic resonance quantum computer.

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In 1982, Feynman [1] observed that quantum-mechanical systems have an information-processing capability much greater than that of classical systems, and could thus potentially be used to implement a new type of powerful computer. Three years later Deutsch [2] described a quantum-mechanical Turing machine, showing that quantum computers could indeed be constructed. Although the theory is well understood, actually building a quantum computer has proved extremely difficult. Up to now, only three methods have been used to demonstrate quantum logical gates: trapped ions [3], cavity QED [4] and NMR [5]. Of these methods, NMR has been the most successful with realizations of quantum teleportation [6], quantum error correction [7], quantum simulation [8], quantum algorithms [9], and others [10]. In this Letter, we add game theory [11] to the list: Quantum games can be experimentally realized on a nuclear magnetic resonance quantum computer.

Recently a new application of quantum information to game theory has been discovered [12–17]. Game theory is an important branch of applied mathematics. It is the theory of decision making and conflict between different agents. Since the seminal book by von Neumann and Morgenstern [18], modern game theory has found applications ranging from economics through biology [19,20]. In the process of a game, whenever a player passes his decision to other players or the game's arbiter, he communicates information. Therefore it is natural to consider the generalization when the information is quantum, rather than classical [12,13]. It should also be noted that many problems in quantum information theory can be considered as quantum games, for instance, quantum cloning [21], quantum cryptography [22], and quantum algorithms [13].

The prisoner's dilemma is a famous game in classical game theory and has been extended into quantum domain by Eisert *et al.* [12]. Their work was based on the maximally entangled state. In this Letter, we generalize the quantum prisoner's dilemma to the case where the players share nonmaximally entangled states. We show that the game exhibits an intriguing structure as a function of the amount of entanglement. In addition, we have realized this quantum game on our nuclear magnetic resonance quantum computer. We believe that it is the first explicit physical realization of such a quantum game.

Let us now briefly recall the quantum prisoner's dilemma presented in Ref. [12]. There are two players, and the players have two possible strategies: cooperate ($\hat{C}$) and defect ($\hat{D}$). The payoff table for the players is shown in Table I. Classically the dominant strategy for both players is to defect (the Nash equilibrium) since no player can improve his/her payoff by unilaterally changing his/her own strategy, even though the *Pareto optimal* is for both players to cooperate. This is the dilemma. In the quantum version, (see Fig. 1), one starts with the product state $|C\rangle|C\rangle$. One then acts on the state with the entangling gate $\hat{J}$ to obtain $|\psi_i\rangle = \hat{J}|CC\rangle = (|CC\rangle + i|DD\rangle)/\sqrt{2}$. The players now act with local unitary operators $\hat{U}_A$ and $\hat{U}_B$ on their qubit. Finally, the disentangling gate $\hat{J}^+$ is carried out and the system is measured in the computational basis, giving rise to one of the four outcomes $|CC\rangle$, $|CD\rangle$, $|DC\rangle$, and $|DD\rangle$. If $\hat{U}_A$ and $\hat{U}_B$ are restricted to the classical strategy space ($\hat{C} = \hat{I}, \hat{D} = i\hat{\sigma}_y$), one then recovers the classical game. If one allows quantum strategies of the form

$$\hat{U}(\theta,\phi) = \begin{pmatrix} e^{i\phi}\cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & e^{-i\phi}\cos\theta/2 \end{pmatrix}, \qquad (1)$$

with $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant \pi/2$, then there exists a new Nash equilibrium, labeled $\hat{Q} \otimes \hat{Q}$, with the payoff $(3,3)$. It has the property of being *Pareto optimal*, therefore the dilemma that exists in the classical game is

TABLE I. Payoff matrix for the prisoner's dilemma. The first entry in the parentheses denotes the payoff of Alice and the second of Bob.

	Bob: $\hat{C}$	Bob: $\hat{D}$
Alice: $\hat{C}$	(3, 3)	(0, 5)
Alice: $\hat{D}$	(5, 0)	(1, 1)

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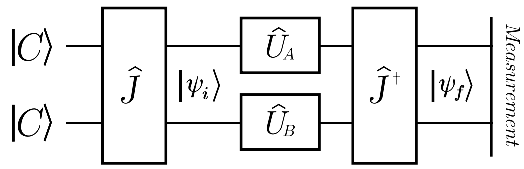

FIG. 1. The setup for the two-player quantum game.

resolved. It was pointed out in Ref. [12] that, if one allows any local operations, then there is no longer a unique Nash equilibrium.

In this present Letter we generalize the Eisert *et al.* scheme by taking the entangling operation to have the form $|\psi_i\rangle = \hat{J}|CC\rangle = \cos(\gamma/2)\,|CC\rangle + i\sin(\gamma/2)\,|DD\rangle$, where $\gamma \in [0, \pi/2]$ measures the entanglement of the initial state. We shall restrict ourselves to strategies of the form of Eq. (1). We will show that an intriguing structure emerges as γ is varied from 0 (no entanglement) to $\pi/2$ (maximally entanglement); namely, the game has two thresholds, $\gamma_{\text{th}1} = \arcsin\sqrt{1/5}$ and $\gamma_{\text{th}2} = \arcsin\sqrt{2/5}$. Figure 2 indicates Alice's expected payoff for $\gamma = \gamma_{\text{th}1}/2$. In this case the game has features similar to the separable game with $\gamma = 0$ (see Ref. [12]). Indeed for $0 \leqslant \gamma \leqslant \gamma_{\text{th}1}$, the quantum game behaves "classically," i.e., the only Nash equilibrium is $\hat{D} \otimes \hat{D}$ and the payoffs for the players are both 1, which is the same as in the classical game. Figure 3 shows Alice's expected payoff with $\gamma = (\gamma_{\text{th}1} + \gamma_{\text{th}2})/2$. Assuming Bob chooses $\hat{D} = \hat{U}(\pi, 0)$, Alice's best strategy is $\hat{Q} = \hat{U}(0, \pi/2)$ with $\$_A(\hat{Q}, \hat{D}) = 5\sin^2\gamma$; while assuming Bob's strategy is $\hat{Q}$, Alice's optimal reply is $\hat{D}$ with $\$_A(\hat{D}, \hat{Q}) = 5\cos^2\gamma$. Since the game is symmetric, the same holds for Bob. Thus, $\hat{D} \otimes \hat{D}$ is no longer a Nash equilibrium because each player can improve his/her payoff by unilaterally deviating from the strategy $\hat{D}$. However, two new Nash equilibria $\hat{Q} \otimes \hat{D}$ and $\hat{D} \otimes \hat{Q}$ appear. This feature holds for $\gamma_{\text{th}1} < \gamma < \gamma_{\text{th}2}$. Indeed, $\$_A[\hat{U}(\theta, \phi), \hat{D}] = \sin^2(\theta/2) + 5\cos^2(\theta/2)\sin^2\phi \times \sin^2\gamma$ and $\$_A[\hat{U}(\theta, \phi), \hat{Q}] = 4 - \cos\theta + [-3 + 2\cos\theta - \cos^2(\theta/2)\cos 2\phi]\sin^2\gamma$; hence $\$_A[\hat{U}(\theta, \phi), \hat{D}] \leqslant 5 \times \sin^2\gamma = \$_A(\hat{Q}, \hat{D})$ and $\$_A[\hat{U}(\theta, \phi), \hat{Q}] \leqslant 5\cos^2\gamma = \$_A(\hat{D}, \hat{Q})$ for all $\theta \in [0, \pi]$ and $\phi \in [0, \pi/2]$, respectively. Analogously $\$_B(\hat{D}, \hat{U}_B) \leqslant \$_B(\hat{D}, \hat{Q}) = 5\sin^2\gamma$ and $\$_B(\hat{Q}, \hat{U}_B) \leqslant \$_B(\hat{Q}, \hat{D}) = 5\cos^2\gamma$ for all $\hat{U}_B$. So $\hat{D} \otimes \hat{Q}$ and $\hat{Q} \otimes \hat{D}$ are both Nash equilibria, with the feature that the payoff for the player who adopts strategy $\hat{D}$ is better than that of the player who adopts $\hat{Q}$. Thus, in this regime the quantum game does not resolve the dilemma. But for $\gamma > \gamma_{\text{th}2}$ quantum strategies resolve the dilemma. In Fig. 4 we depict Alice's payoff as a function of the strategies $\hat{U}_A$ and $\hat{U}_B$ with $\gamma = (\gamma_{\text{th}2} + \pi/2)/2$. This figure is similar to the one for the maximally entangled game in Ref. [12]. It can be shown that $\hat{Q} \otimes \hat{Q}$ is a unique equilibrium not only for $\gamma = (\gamma_{\text{th}2} + \pi/2)/2$ but also for any $\gamma \in [\gamma_{\text{th}2}, \pi/2]$. Hence a novel Nash equilibrium $\hat{Q} \otimes \hat{Q}$ arises with payoff $\$_A(\hat{Q}, \hat{Q}) = \$_B(\hat{Q}, \hat{Q}) = 3$, which has the property of being *Pareto optimal* [19]. The dilemma that exists in the classical game is removed as long as the game's entanglement exceeds the threshold $\gamma_{\text{th}2} = \arcsin\sqrt{2/5} \approx 0.685$, even though the game's initial state is not maximally entangled.

Figure 5 indicates Alice's payoff as a function of the parameter γ when both players resort to the Nash equilibrium. The two thresholds are analogous to phase transitions. When the amount of entanglement is less than the smaller threshold, one is in a classical region. When the amount of entanglement lies between the two thresholds, one is in a transition region between classical and quantum

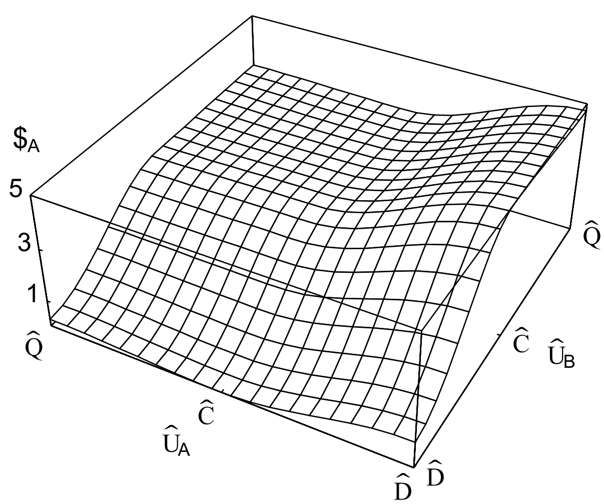

FIG. 2. Alice's payoff for $\gamma = \gamma_{\text{th}1}/2$. In this and the following two plots, we have chosen a parametrization such that the strategies $\hat{U}_A$ and $\hat{U}_B$ each depend on a single parameter $t \in [-1, 1]$: $\hat{U}_A = \hat{U}(t\pi, 0)$ for $t \in [0, 1]$ and $\hat{U}_A = \hat{U}(0, -t\pi/2)$ for $t \in [-1, 0]$ (same for Bob). Cooperation $\hat{C}$ corresponds to the value $t = 0$, defection $\hat{D}$ to $t = 1$, and $\hat{Q}$ to $t = -1$.

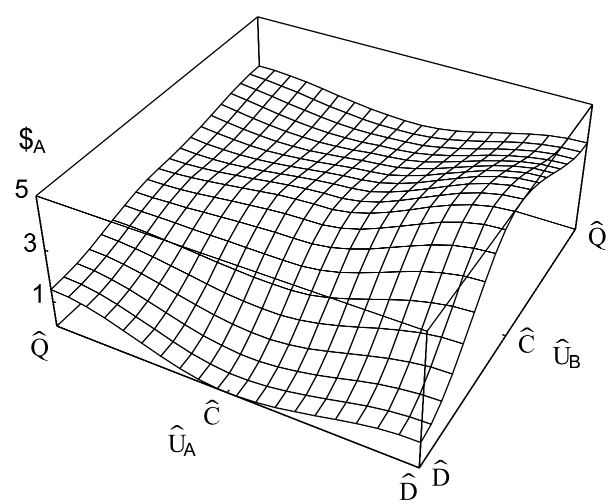

FIG. 3. Alice's payoff for $\gamma = (\gamma_{\text{th}1} + \gamma_{\text{th}2})/2$. The parametrization is chosen as in Fig. 2.

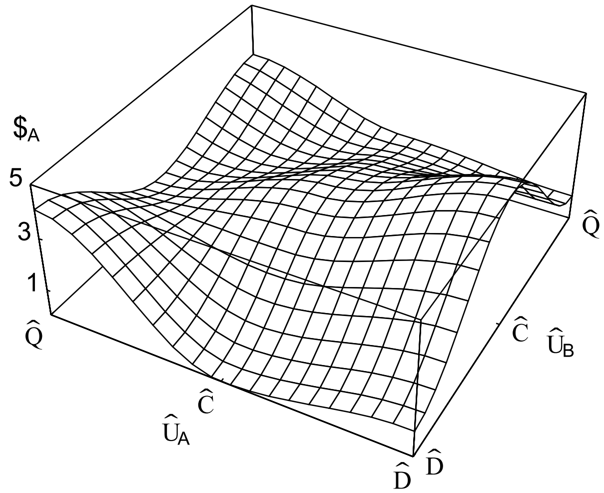

FIG. 4. Alice's payoff for $\gamma = (\gamma_{\mathrm{th}2} + \pi/2)/2$. The parametrization is the same as Fig. 2.

behavior. The last domain is the fully quantum region. It is surprising that, in the transition region, both Nash equilibria result in an unfair game, even though the structure of the game is symmetric with respect to the interchange of the two players. We think that the reasons for the asymmetry are as follows: (i) Since the definition of Nash equilibrium allows multiple Nash equilibria to coexist, the solutions may be degenerated. Therefore the definition itself allows the possibility of such an asymmetry. This situation is similar to spontaneous symmetry breaking. (ii) If we consider the two Nash equilibria as a whole, they are fully equivalent and the game remains symmetric. But, finally, the two players have to choose one from the two equilibria. This also causes the asymmetry of the game.

This quantum game was implemented using our two qubit NMR quantum computer, described in Ref. [23].

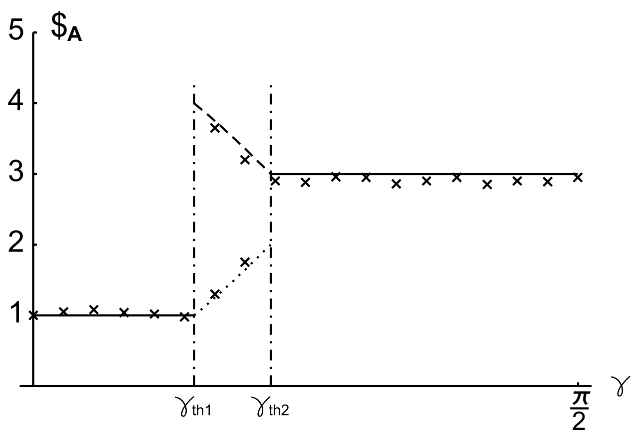

FIG. 5. The expected payoff for Alice as a function of the measure of the parameter γ when both players resort to Nash equilibrium. The line corresponds to theoretic calculation and the crosses to the experimental results. For $\gamma_{\mathrm{th}1} < \gamma < \gamma_{\mathrm{th}2}$, the dashed line (dotted line) represents Alice's payoff when the Nash equilibrium is $\hat{D} \otimes \hat{Q}$ ($\hat{Q} \otimes \hat{D}$).

This computer uses the two spin states of ^{1}H nuclei of partially deuterated cytosine in a magnetic field as qubits, while radio frequency (rf) fields and spin-spin couplings between the nuclei $J_{AB} = 7.17$ Hz are used to implement quantum logic gates. Experimentally, we performed nineteen separate sets of experiments with the entanglement of the player's qubits given by $\gamma = n\pi/36$ ($n = \{0, 1, 2, \ldots, 18\}$). The $\gamma = 0$ ($n = 0$) corresponds to the Eisert *et al.* separable game, and $\gamma = \pi/2$ ($n = 18$) corresponds to their maximally entangled quantum game. In each set, the full process of the quantum game shown in Fig. 1 was executed. The details of the process are as follows: (i) The quantum game starts with the computer in the unentangled pure state $|CC\rangle$, but with an NMR quantum computer it is impossible to begin in a true pure state. Using the methods of Cory *et al.* [24] it is, however, possible to create an effective pure state, which behaves in an equivalent manner. (ii) The initial entangled state is obtained by applying the entangling gate $\hat{J} = \exp\{i\gamma\hat{D} \otimes \hat{D}/2\}$ which was performed with the pulse sequence shown in Fig. 6, where the time period $t = \gamma/(\pi J_{AB})$. (iii) Players Alice and Bob execute their strategic moves (the Nash equilibrium) described as local unitary operations $\hat{U}_A \otimes \hat{U}_B$. As shown above, $\hat{U}_A \otimes \hat{U}_B$ is determined by the value of $\gamma = \pi J t = n\pi/36$. Experimentally, $\hat{D} \otimes \hat{D}$ ($0 \leqslant \gamma < \gamma_{\mathrm{th}1}$, i.e., $n = \{0, 1, 2, 3, 4, 5\}$) was implemented using a nonselective 180°_y pulse; $\hat{D} \otimes \hat{Q}$ ($\hat{Q} \otimes \hat{D}$) ($\gamma_{\mathrm{th}1} \leqslant \gamma \leqslant \gamma_{\mathrm{th}2}$, i.e., $n = \{6, 7\}$) was implemented by performing a selective 180°_y pulse on Alice's (Bob's) qubit, while a selective pulse sandwich $90^\circ_{-y} - 180^\circ_x - 90^\circ_y$ was performed on Bob's (Alice's) qubit; and $\hat{Q} \otimes \hat{Q}$ ($\gamma_{\mathrm{th}2} \leqslant \gamma \leqslant \pi/2$, i.e., $n = \{8, 9, \ldots, 18\}$) was implemented using a composite nonselective pulse sandwich $90^\circ_{-y} - 180^\circ_x - 90^\circ_y$. (iv) Finally, the disentangling gate $\hat{J}^+ = \exp\{-i\gamma\hat{D} \otimes \hat{D}/2\}$ (the inverse of $\hat{J}$) is applied before the measurement. The pulse sequence to implement $\hat{J}^+$ is the same as in Fig. 6, except for $t = (2\pi - \gamma)/(\pi J_{AB})$. Thus the final state $|\psi_f\rangle = |\psi_f(\hat{U}_A, \hat{U}_B)\rangle$ of the game prior to measurement is given by $|\psi_f\rangle = \hat{J}^+(\hat{U}_A \otimes \hat{U}_B)\hat{J}|CC\rangle$.

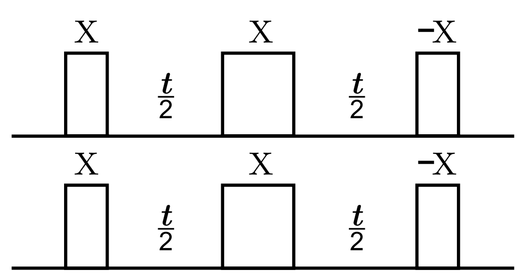

FIG. 6. NMR pulse sequence used to implement the entangling gate J. The narrow boxes correspond to 90° pulses, whereas wide boxes are 180° pulses; the upper and lower lines refer to the nuclear spins corresponding to Alice's and Bob's qubits, respectively; the phase of each pulse is written above it.

In NMR experiments, it is not practical to determine the final state directly, but an equivalent measurement can be made by so-called quantum state tomography to recover the density matrix $\rho = |\Psi_f\rangle\langle\Psi_f|$ [5]. Then the expected payoff was determined using the numerical values of the payoff table of prisoner's dilemma by the $\$_A = 3P_{CC} + 5P_{DC} + P_{DD}$ and $\$_B = 3P_{CC} + 5P_{CD} + P_{DD}$, where $P_{\sigma\sigma'} = \langle\sigma\sigma'|\rho|\sigma\sigma'\rangle$ is the probability of finding the eigenstate $|\sigma\sigma'\rangle$ (with $\sum_{\sigma,\sigma'\in\{C,D\}} P_{\sigma\sigma'} = 1$).

All experiments were conducted at room temperature and pressure on the Bruker Avance DMX-500 spectrometer in the Laboratory of Structure Biology, University of Science and Technology of China. Alice's payoffs as a function of the parameter γ (the measure of entanglement) in our NMR experiments are shown in Fig. 5. The computations shown in Fig. 1 took less than 300 ms, which was well within the decoherence time $T_2 \approx 3$ s. The relationship between the player's payoff and the parameter γ in the quantum game is clearly seen in Fig. 5, with good agreement between theory and experiment. The relative error is less than 8%. The errors are primarily due to inhomogeneity of the magnetic field, imperfect 90° and 180° pulses, and the variability over time of the measurement process.

In summary, it was shown in Ref. [12] that the classical prisoner's dilemma can be generalized into a quantum game, and that when a maximally entangled state is employed the dilemma disappears. We used the same physical model as Eisert *et al.,* but introduced a new parameter γ, which measures the amount of entanglement in the quantum game. As γ varies, novel features appear: there are two thresholds, $\gamma_{\text{th}1}$ and $\gamma_{\text{th}2}$, which separate the classical region, an intermediate region where two Nash equilibrium coexist, and a fully quantum region where the dilemma disappears. The fact that the dilemma can be removed as long as the game's entanglement exceeds a certain threshold $\gamma_{\text{th}2}$ is very much as in quantum cryptography and computation, where the superior performance of the quantum system depends strongly on the amount of entanglement. Furthermore, we realized this scheme experimentally on our two-qubit ensemble quantum computer. These experimental results demonstrate how a NMR quantum computer can load an initial state, enable each player to perform his/her quantum strategic moves, and readout the payoffs. This reveals a new domain of application for quantum computers.

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Note added.—Since this work was carried out we have generalized it in three ways: First, we have considered the correlations between entanglement and quantum games for different sets of strategies [17]. Second, we have considered three-player entanglement enhanced quantum games [25]. Finally, we analyzed how the thresholds $\gamma_{\text{th}1}$, $\gamma_{\text{th}2}$ vary when the parameters in the payoff table are changed [26].

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[1] R. Feynman, Int. J. Theor. Phys. **21**, 467 (1982).
[2] D. Deutsch, Proc. R. Soc. London A **400**, 97 (1985).
[3] J. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[4] C. Monroe *et al.,* Phys. Rev. Lett. **75**, 4714 (1995).
[5] D. G. Cory, A. F. Fahmy, and T. F. Havel, Proc. Natl. Acad. Sci. U.S.A. **94**, 1634 (1997); N. Gershenfeld and I. Chuang, Science **275**, 350 (1997).
[6] M. A. Nielsen, E. Knill, and R. Laflamme, Nature (London) **396**, 52 (1998).
[7] D. G. Cory *et al.,* Phys. Rev. Lett. **81**, 2152 (1998); D. Leung *et al.,* Phys. Rev. A **60**, 1924 (1999); E. Knill *et al.,* Phys. Rev. Lett. **86**, 5811 (2001).
[8] S. Somaroo *et al.,* Phys. Rev. Lett. **82**, 5381 (1998).
[9] I. L. Chuang *et al.,* Phys. Rev. Lett. **80**, 3408 (1998); I. L. Chuang *et al.,* Nature (London) **393**, 143 (1998); J. A. Jones, M. Mosca, and R. H. Hansen, Nature (London) **393**, 344 (1998); J. A. Jones and M. Mosca, Phys. Rev. Lett. **83**, 1050 (1999); R. Marx *et al.,* Phys. Rev. A **62**, 012310 (2000); L. M. K. Vandersypen *et al.,* Phys. Rev. Lett. **85**, 5452 (2000); J. Du *et al.,* Phys. Rev. A **64**, 042306 (2001).
[10] X. Fang, X. Zhu, M. Feng, X. Mao, and F. Du, Phys. Rev. A **61**, 022307 (2000); J. Du *et al.,* Phys. Rev. A **63**, 042302 (2001).
[11] For an introduction, see, e.g., E. Rasmusen, *Games and Information* (Blackwell, Oxford, UK, 1995).
[12] J. Eisert, M. Wilkens, and M. Lewenstein, Phys. Rev. Lett. **83**, 3077 (1999).
[13] D. A. Meyer, Phys. Rev. Lett. **82**, 1052 (1999); D. A. Meyer, quant-ph/0004092.
[14] L. Goldenberg, L. Vaidman, and S. Wiesner, Phys. Rev. Lett. **82**, 3356 (1999).
[15] S. C. Benjamin and P. M. Hayden, Phys. Rev. A **64**, 030301 (2001); N. F. Johnson, Phys. Rev. A **63**, 020302 (2001).
[16] A. Iqbal and A. H. Toor, Phys. Lett. A **280**, 249 (2001); C. F. Li *et al.,* Phys. Lett. A **280**, 257 (2001).
[17] J. Du *et al.,* Phys. Lett. A **289**, 9 (2001).
[18] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior,* third edition (Princeton University Press, Princeton, NJ, 1953).
[19] R. B. Myerson, *Game Theory: Analysis of Conflict* (Harvard University Press, Cambridge, MA, 1991).
[20] M. A. Nowak and K. Sigmund, Nature (London) **398**, 367 (1999).
[21] R. F. Werner, Phys. Rev. A **58**, 1827 (1998).
[22] A. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[23] J. Du *et al.,* Chin. Phys. Lett. **17**, 64 (2000).
[24] D. G. Cory, M. D. Price, and T. F. Havel, Physica (Amsterdam) **120D**, 82 (1998).
[25] J. Du *et al.,* quant-ph/0110122.
[26] J. Du *et al.,* quant-ph/0111138.